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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES: PART IV

by

RONALD W. SHEPHARD

and

ROKAYA A. AL-AYAT

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DYNAMIC THEORY OF PRODUCTION CORRESPONDENCES†

PART IV

by

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ABSTRACT

Chapter 10 of a monograph on a Dynamic Theory of Production Correspondences is presented. A network of correspondences is defined and properties are investigated. General technical coefficient form of network correspondences are introduced and several special structures are considered with algorithms for computing "greedy" histories of input and output rate histories.

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CHAPTER 10

DYNAMIC ACTIVITY ANALYSIS*

In the previous chapters the dynamic model of production has been abstract. For application of the concepts developed, the structure of production has to be made more explicit. The most direct way of doing this is to consider production as being carried out by a finite number of mutually supporting activities. In general form, each activity may be taken as an abstract production correspondence in order to investigate the structure of a production network. This structure can be made more specific for application by taking the production structure of each activity to be represented by technical coefficients which can be allowed to vary in time for such phenomena as "learning," or programmed technological change. It will be seen that in this way a dynamic production correspondence may be formulated for such widely different production activities as construction (e.g., shipbuilding) and job shop manufacture.

10.1 General Form of a Production Network Output Correspondence

Let N denote the number of distinct activities in a production system. For the purposes of the treatment in this section these activities are taken as primitive elements. Denote them by $A_1, A_2, \ldots, A_N$. An initial activity A_o is added as a source of exogenous inputs, and a terminal activity A_{N+1} is included to collect all end products. The outputs of an activity may be intermediate or final, or both kinds of production. Spare parts are an example of the latter. All output

*This chapter is in part a revision and extension of R. W. Shephard, R. Al-Ayat and R. C. Leachman's "Shipbuilding Production Function," in QUANTITATIVE WIRTSCHAFTSFORSCHUNG, Mohr, Tubingen, 1977.

histories, whatever the purpose of the product, are here taken as elements of a vector of a common function space $(L_\infty)^m_+$, where m denotes the total number of all kinds of products.

For notation:

$x \varepsilon (L_\infty)^n_+$ is a vector of histories of available exogenous input rates. (n inputs.)

$x_{oi} \varepsilon (L_\infty)^n_+$ is a vector of time histories of allocations of x to the production activity A_i $(i = 1,2, \ldots, N)$.

$V_i \varepsilon (L_\infty)^m_+$ is a vector of histories of net output rates provided by the production activity A_i $(i = 1,2, \ldots, N)$.

$V_{ij} \varepsilon (L_\infty)^m_+$ is a vector of time histories of input rates to A_j of outputs of A_i $(i = 1,2, \ldots, N)$, $(j = 1,2, \ldots, N)$, as intermediate product input, and $V_{i,N+1}$ is a time history of net output from A_i .

$v_i := \left(\sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon (L_\infty)^m_+$ is a vector of time histories of total input rates of intermediate products to A_i $(i = 1,2, \ldots, N)$. By $\left(\sum'\right)$ is denoted a sum with V_{ii} excluded. The vector V_i of time histories of output rates of A_i is taken as net of V_{ii} .

With the foregoing notation, the dynamic production correspondence of A_i $(i = 1,2, \ldots, N)$ is expressed by

$$(10.1\text{-}1) \qquad (x_{oi}, v_i) \varepsilon (L_\infty)^n_+ \times (L_\infty)^m_+ \to \mathbb{P}_i(x_{oi}, v_i) \varepsilon 2^{(L_\infty)^m_+} .$$

The input rate histories v_i are exogenous to A_i .

For the activity A_o

$$(10.1\text{-}2) \qquad \sum_{i=1}^{N} x_{oi} \leqq x$$

is an assumption of strong disposal of input histories for the production network. In the case of the activity A_{N+1} which merely collects the net outputs,

$$(10.1\text{-}3) \qquad V_{N+1} := \sum_{i=1}^{N} V_{i,N+1} \ .$$

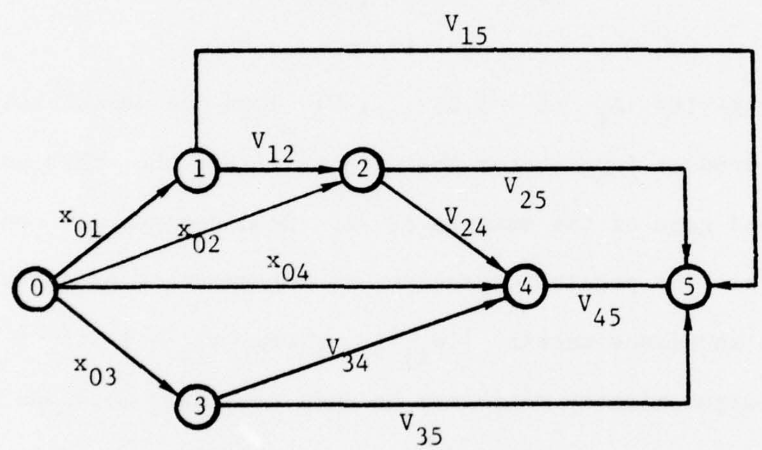
A producing activity A_i $(i = 1,2, \ldots, N)$ does not necessarily receive intermediate product inputs from the outputs of all the other producing activities, and none of the outputs of A_i need necessarily contribute to net outputs. The possible transfers of intermediate products are defined by an incidence matrix $||w_{ij}||$, where $w_{ij} = 1$ if V_i contains an output history which may be used to supply intermediate product input to A_j , $j \neq i$, $j \neq N+1$, or final output for A_{N+1} , and otherwise $w_{ij} = 0$, $(i = 1,2, \ldots, N)$, $(j = 1,2, \ldots, N)$. If A_i sends output to A_{N+1} , $w_{i,N+1} = 1$ and otherwise $w_{i,N+1} = 0$.

Assumption of Essentiality of Exogenous Inputs: $\mathbb{P}.\mathbb{N}.1$

(10.1-4) The exogenous inputs for an activity A_i are globally essential and input rate strongly limitational for V_i , $(i = 1,2, \ldots, N)$.

The output of an activity may be intermediate product entirely, or final product entirely, or a combination of both. It must be possible that $V_{i,N+1}$ is positive in some component on a subset of positive measure for some $(i = 1,2, \ldots, N)$, otherwise the production system may not yield any net output.

As a simple illustration, consider four producing activities represented schematically by nodes (1), (2), (3), (4) as illustrated below:



The directed arcs (12), (24) and (34) show the transfers of intermediate product inputs. To this system has been added a node (0) as source for exogenous input flows, and a node (5) to collect net output flows. The directed arcs (01), (02), (03), (04) show exogeneous input flows to the producing nodes, and the directed arcs (15), (25), (35) and (45) show possible net output flows. These two sets of arcs are labelled in accordance with the notation defined above. Obviously such a directed graph representation of the activity structure of a production system can become quite complicated.

Returning now to the general treatment, the output correspondence for the entire activity structured production network is denoted as before by

$$(10.1\text{-}5) \qquad x \; \varepsilon \; (L_\infty)^n_+ \rightarrow \mathbb{P}(x) \; \varepsilon \; 2^{(L_\infty)^m_+}$$

where $\mathbb{P}(x)$ may take several forms depending upon assumptions made as to disposability of outputs and inputs for the correspondences (10.1-1) of the activities A_i $(i = 1,2, \ldots, N)$. In the case of free disposability of inputs and outputs

$$(10.1\text{-}6) \qquad \begin{aligned} \mathbb{P}(x) := \Bigg\{ u \; \varepsilon \; (L_\infty)^m_+ : u \leqq \sum_{i=1}^{N} V_{i,N+1}, \left(\sum_{j=1}^{N+1}{}' \, V_{ij} \right) \varepsilon \; \mathbb{P}_i \left(x_{oi}, \sum_{j=1}^{N}{}' \, V_{ji} \right) , \\ (i = 1,2, \ldots, N) \; , \; \sum_{i=1}^{N} x_{oi} \leqq x \Bigg\} \, . \end{aligned}$$

This definition of the map set $\mathbb{P}(x)$ of net output rate histories for the entire production system is modified for scaled disposal of output rate histories by changing the first constraint to

$$(10.1\text{-}7) \quad u_\alpha = \theta_\alpha \sum_{i=1}^{N} (V_{i,N+1})_\alpha \; , \; \theta_\alpha \; \varepsilon \; [0,1] \; , \; (\alpha = 1,2, \ldots, m) \; ,$$

for which $(\;)_\alpha$ denotes the α-th component. In this case the correspondences (10.1-1) for the activities $A_1, \ldots, A_N$ may or may not be taken to obey property $\mathbb{P}$.6S. Further when $\theta_\alpha = \theta \; \varepsilon \; [0,1]$ for $(\alpha = 1,2, \ldots, m)$, a weak disposal (common scaling) of output rate histories is obtained, irrespective of the output disposal assumption for the activities A_i .

In the case of scaled disposal of exogenous input rate histories, the definition of the map set $\mathbb{P}(x)$ in (10.1-6) is modified by substituting

$$(10.1\text{-}8) \quad \sum_{i=1}^{N} (x_{oi})_j = \frac{1}{\lambda_j} (x)_j \ , \ \lambda_j \in [1,+\infty) \ , \ (j = 1,2, \ldots, n) \ ,$$

for the last constraint. Property $\mathbb{P}$.3S may or may not be invoked for the correspondences (10.1-1). When $\lambda_j = \lambda \in [1,+\infty)$, $(j = 1,2, \ldots, n)$, a weak disposal of input rate histories is expressed; but the input disposal assumptions for the activities is not circumscribed.

The various forms of disposal of input and output rate histories may be freely combined for the definition of $\mathbb{P}(x)$. The disposal of intermediate product outputs is determined by the assumptions made for the correspondences (10.1-1) independently of those for the network.

No explicit allowance has been made for storage of activity outputs (as intermediate products) or exogenous inputs. Provision for such storage in the definition of the correspondence (10.1-5), (10.1-6) may be made as follows:

(a) Define cumulative histories $(\hat{V}_{ij})_k$ of output rates of activity A_i used as input to A_j by

$$(10.1\text{-}9) \qquad (\hat{V}_{ij}(t))_k := \int_0^t (V_{ij}(\tau))_k d\nu_k(\tau) \qquad t \in [0,+\infty)$$

for $(k = 1,2, \ldots, m)$ and $(i = 1,2, \ldots, N)$, $(j = 1,2, \ldots, (N + 1))$.

(b) Define cumulative histories $(\hat{x}_{oi})_j$ of exogenous input rate allocations by:

$$(10.1\text{-}10) \qquad (\hat{x}_{oi}(t))_j := \int_0^t (x_{oi}(\tau))_j d\mu_j(\tau) \ , \ t \in [0,+\infty)$$

for $(j = 1,2, \ldots, n)$ and $(i = 1,2, \ldots, N)$.

(c) Let $\hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)\right)$ denote for $(i = 1,2, \ldots, N)$ the cumulative output histories of A_i obtained from $\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)$, defined by:

$$\hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)\right) := \Big\{\hat{V}_i : \hat{V}_i = ((\hat{V}_i)_1, (\hat{V}_i)_2, \ldots, (\hat{V}_i)_m)$$

$$(10.1\text{-}11) \qquad (\hat{V}_i(t))_k := \int_0^t (V_i(t))_k d\nu_k(\tau) \ , \ t \in [0,+\infty) \ , \ (k = 1,2, \ldots, m) \ ,$$

$$V_i := ((V_i)_1, (V_i)_2, \ldots, (V_i)_m) \ ; \ V_i \in \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)\Big\} \ ,$$

$$(i = 1,2, \ldots, N) \ .$$

(d) Modify the second constraint of (10.1-6) to

$$(10.1\text{-}12) \qquad \left(\sum_{j=1}^{N+1}{}' \hat{V}_{ij}\right) \in \hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right)\right) \quad (i = 1,2, \ldots, N) \ .$$

(e) Let $\{\sigma_1, \sigma_2, \ldots, \sigma_S\} \subset \{1,2, \ldots, n\}$ denote exogenous input indices for storable inputs, and define cumulative availability of these inputs by

$$(10.1\text{-}13)\quad (\hat{x}(t))_{\sigma_j} := \int_0^t (x(\tau))_{\sigma_j}\, d\mu_{\sigma_j}(\tau)\ ,\ t \in [0,+\infty)\ ,\ (j = 1,2,\ \ldots,\ S)\ .$$

(f) Modify the last constraint of (10.1-6) to

$$(10.1\text{-}14)\quad \begin{aligned} &\sum_{i=1}^{N} (\hat{x}_{oi})_{\sigma_j} \leqq (\hat{x})_{\sigma_j} && (j = 1,2,\ \ldots,\ S) \\ &\sum_{i=1}^{N} (x_{oi})_{\sigma_j} \leqq (x)_{\sigma_j} && (j = (S+1),(S+2),\ \ldots,\ n)\ . \end{aligned}$$

With these modifications, the map sets of the production network output correspondence for free disposal of input and output rate histories becomes:

$$(10.1\text{-}15)\quad \begin{aligned} \mathbb{P}(x) := \Bigg\{ u \in (L_\infty)_+^m : u \leqq \sum_{i=1}^{N} V_{i,N+1},\ \left(\sum_{j=1}^{N+1}{}' \hat{V}_{ij}\right) \in \hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(x_{oi},\ \sum_{j=1}^{N} V_{ji}\right)\right), \\ (i = 1,2,\ \ldots,\ N)\ ,\ \sum_{i=1}^{N} (\hat{x}_{oi})_{\sigma_j} \leqq (\hat{x})_{\sigma_j}\ ,\ (j = 1,2,\ \ldots,\ S)\ , \\ \sum_{i=1}^{N} (x_{oi})_{\sigma_j} \leqq (x)_{\sigma_j}\ ,\ (j = (S+1),(S+2),\ \ldots,\ n) \Bigg\}\ . \end{aligned}$$

Modification of (10.1-15) for strong and weak disposal of output rate histories is made by introducing (10.1-7) for the first constraint as before. In the case of strong or weak disposal of input rate histories, one replaces the last two constraints of (10.1-15) by

$$\sum_{i=1}^{N} (\hat{x}_{oi})_{\sigma_j} = \frac{1}{\lambda_{\sigma_j}} (\hat{x})_{\sigma_j}\ ,\ (j = 1,2,\ \ldots,\ S)$$

$$\sum_{i=1}^{N} (x_{oi})_{\sigma_j} = \frac{1}{\lambda_{\sigma_j}} (x)_{\sigma_j} \quad , \; (j = (S+1), (S+2), \ldots, n)$$

with $\lambda_{\sigma_j} \varepsilon [1,+\infty)$, for $(j = 1,2, \ldots, n)$, and in case of weak disposability set $\lambda_{\sigma_j} = \lambda \; \varepsilon \; [1,+\infty)$, $(j = 1,2, \ldots, n)$.

In any practical case the constraints (10.1-12) and (10.1-14) need some upper bounds on the net accumulations of intermediate products and exogenous inputs. But this is a matter of detail not considered at this level of abstraction.

10.2 Properties of the Production Network Output Correspondence

It is of some interest to investigate the properties of a production network implied by the axioms for the dynamic production correspondences of the activities of the network.

If $x = 0$, the constraints (10.1-2), (10.1-8) or (10.1-14) all imply $x_{oi} = 0$ for $(i = 1,2, \ldots, n)$. Consequently by (10.1-4) it follows for (10.1-6) that

$$\sum_{j=1}^{N+1}{}' \, V_{i,j} = 0 \; , \; (i = 1,2, \ldots, N)$$

and $\sum_{i=1}^{N} V_{i,N+1} = 0$, whence $\mathbb{P}(x) = \{0\}$ whether or not (10.1-7) is used. If the contraint (10.1-14) is used in the case of inventories, the same result follows. Hence axiom $\mathbb{P}.1$ may be taken as a generally applicable property for production networks. See Chapter 3, Section 3.1 for the implication of the assumption $\mathbb{P}.\mathbb{N}.1$, $(i = 1,2, \ldots, N)$.

Concerning property $\mathbb{P}.2$ for the correspondence (10.1-5), (10.1-6), if the vector x of exogenous input rate histories is bounded (in the norm), the constraints (10.1-2), (10.1-8) or (10.1-14) imply that the vectors x_{oi}, $(i = 1,2, \ldots, N)$, are likewise bounded. Then, since x_{oi} spans an essential subset of factors for the correspondence (10.1-1) which are globally essential and input rate strongly limitational for vectors V_i by $\mathbb{P}.\mathbb{N}.1$, it follows that the sets $\mathbb{P}_i(x_{oi}, v_i)$ are likewise bounded. Consequently, $\left|\left| \sum_{j=1}^{N+1}{}' V_{ij} \right|\right|$ is bounded, implying $\left(\sum_{i=1}^{N} V_{i,N+1} \right)$ is bounded. Hence $\mathbb{P}(x)$ is bounded. Thus property $\mathbb{P}.2$ holds for the production network in all cases, including inventories. See (10.1-15).

In regard to property $\mathbb{P}.3$, if it holds for the correspondence of each activity, it clearly holds for the system correspondence. To see this, let $u \in \mathbb{P}(x)$. Then by (10.1-6) it follows that

$$u \leqq \sum_{i=1}^{N} V_{i,N+1} , \left(\sum_{j=1}^{N+1}{}' V_{ij} \right) \in \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji} \right) , \quad (i = 1,2, \ldots, N)$$

and

$$\sum_{i=1}^{N} x_{oi} \leqq x .$$

Obviously u satisfies this system of inequalities when the last one is modified to

$$\sum_{i=1}^{N} x_{oi} \leqq \lambda x , \quad \lambda \in [1,+\infty) .$$

Consequently, $u \varepsilon \mathbb{P}(\lambda x)$ and $\mathbb{P}(x) \subset \mathbb{P}(\lambda x)$ for $\lambda \varepsilon [1,+\infty)$. The same argument applies when (10.1-8) is used in place of the last inequality of (10.1-6) or (10.1-7) holds, and also when inventories are allowed. See (10.1-15). Thus property $\mathbb{P}.3$, $\mathbb{P}.3S$ or $\mathbb{P}.3SS$ holds for the correspondence of a production network when the correspondences of the activities correspondingly satisfy the same property.

A stronger property than $\mathbb{P}.3SS$ also propagates for the production network. Consider:

Definition (10.1-1):

A production correspondence $x \varepsilon (L_\infty)^n_+ \to \mathbb{P}(x) \varepsilon 2^{(L_\infty)^m_+}$ exhibits nondecreasing returns to scale iff $\mathbb{P}(\lambda x) \supset \lambda \mathbb{P}(x)$, $\lambda \varepsilon [0,+\infty)$.

Then the following proposition holds:

Proposition (10.1-1):

If the correspondences of the activities of a production network exhibit nondecreasing returns to scale, the correspondence of the network has the same property.

Proof of Proposition (10.1-1) runs as follows:

Let

$$u \varepsilon \mathbb{P}(x) = \left\{ u : u \leqq \sum_{i=1}^{N} v_{i,N+1}, \left(\sum_{j=1}^{N+1}{}' v_{ij}\right) \varepsilon \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' v_{ji}\right) , \right.$$

$$\left. (i = 1,2, \ldots, N) , \sum_{i=1}^{N} x_{oi} \leqq x \right\} .$$

Then for $\lambda \varepsilon (0,+\infty)$,

$$u \in \frac{1}{\lambda} \left\{ \lambda u : \lambda u \leqq \sum_{i=1}^{N} \lambda V_{i,N+1}, \left(\sum_{j=1}^{N+1}{}' \lambda V_{ij} \right) \in \lambda \mathbb{P}_i \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji} \right) \right.$$

$$\left. (i = 1,2, \ldots, N) , \sum_{i=1}^{N} (\lambda x_{oi}) \leqq \lambda x \right\} .$$

When the activities A_i $(i = 1, \ldots, N)$ exhibit nondecreasing returns to scale, the right side of the second constraint may be expressed

$$\lambda \mathbb{P}_i \left(x_{oi}, \sum_{i=1}^{N}{}' V_{ij} \right) \subset \mathbb{P}_i \left((\lambda x_{oi}), \sum_{j=1}^{N}{}' \lambda V_{ji} \right)$$

and then $u \in \frac{1}{\lambda} \mathbb{P}(\lambda x)$ or $\mathbb{P}(\lambda x) \supset \lambda \mathbb{P}(x)$. The same argument can be made for any weak disposal of inputs and outputs, i.e., (10.1-7), (10.1-8), and accumulation of inventories since $\hat{\mathbb{P}}_i \left(\lambda \mathbb{P}_i \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji} \right) \right) = \lambda \hat{\mathbb{P}}_i \left(\mathbb{P}_i \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji} \right) \right)$. The case of $\lambda = 0$ may be dismissed.

To continue with the axioms for a dynamic production correspondence, property $\mathbb{P}.4.1$ is not applicable, because the output space $(L_\infty)^m_+$ spans both intermediate and final products.

Concerning property $\mathbb{P}.4.2$, observe that if $\bar{u} \in \mathbb{P}(x)$:

$$\bar{u} \leqq \sum_{i=1}^{N} V_{i,N+1} , \left(\sum_{j=1}^{N+1}{}' V_{ij} \right) \in \mathbb{P}_i \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji} \right) , (i = 1,2, \ldots, N) ,$$

$$\sum_{i=1}^{N} x_{oi} \leqq x .$$

Using property $\mathbb{P}.4.2$ for the activities when $\theta \in (0,+\infty)$, there exist positive scalars $\mu_\theta^{(i)}$, $(i = 1, \ldots, N)$, such that

$$\left(\sum_{j=1}^{N+1}{}' (\theta V_{ij})\right) \varepsilon \ \mathbb{P}_i\left(\mu_\theta^{(i)} x_{oi}, \mu_\theta^{(i)} \sum_{j=1}^{N}{}' V_{ji}\right), \ (i = 1,2, \ldots, N) .$$

Let $\lambda_\theta = \underset{i}{\mathrm{Max}}\ \mu_\theta^{(i)}$. Then, under weak disposal of inputs, i.e., axiom $\mathbb{P}.3$,

$$\theta\bar{u} \leqq \theta \sum_{i=1}^{N+1} V_{i,N+1} \quad \text{with} \quad \theta \sum_{j=1}^{N+1}{}' V_{ij} \ \varepsilon \ \mathbb{P}_i\left(\lambda_\theta x_{oi}, \lambda_\theta \sum_{j=1}^{N}{}' V_{ji}\right),$$

$$(i = 1,2, \ldots, N) ,$$

and

$$\sum_{i=1}^{N} \lambda_\theta x_{oi} \leqq \lambda_\theta x .$$

Hence property $\mathbb{P}.4.2$ holds for the correspondence of the production network when it applies to the correspondences of the activities. Since only a weak disposal of input histories was used, the argument obviously applies when stronger disposals are possible. Also, the argument is not essentially altered in the case of weak disposal of outputs. When inventories are included one need only write

$$\theta \sum_{j=1}^{N+1}{}' \hat{V}_{ij} \ \varepsilon \ \hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(\lambda_\theta x_{oi}, \lambda_\theta \sum_{j=1}^{N}{}' V_{ji}\right)\right) \quad (i = 1,2, \ldots, N)$$

in place of the second constraint and carry out the argument as before. Thus property $\mathbb{P}.4.2$ holds for the production network in all circumstances.

Next, concerning property $\mathbb{P}.5$ for the production network, the generality of the latter is such that this property does not follow from the same property for the correspondences of the activities without

additional assumption. For this purpose it is assumed either that $\mathbb{P}.2S$ and $\mathbb{P}.5$ Bis hold along with property $\mathbb{P}.5$ for the activity correspondences when the norm topology is used for vectors of output histories, or $\mathbb{P}.2$, $\mathbb{P}.5$ Bis and $\mathbb{P}.5$ hold under a weak* topology for vectors of output histories. In either case the correspondences

$$\left(x_{oi},\ \sum_{j=1}^{N}{}' V_{ji}\right) \to \mathbb{P}_i\left(x_{oi},\ \sum_{j=1}^{N}{}' V_{ji}\right) \qquad (i = 1,2,\ \ldots,\ N)$$

have compact map sets so that the conditions for Proposition (2.1.1-1) are fulfilled. Some alteration of the expression of the network correspondence (10.1-5), (10.1-6) will likewise be used. The constraints

$$\left(\sum_{j=1}^{N+1}{}' V_{ij}\right) \varepsilon\ \mathbb{P}_i\left(x_{oi},\ \sum_{j=1}^{N}{}' V_{ji}\right) \qquad (i = 1,2,\ \ldots,\ N)$$

do not lend themselves to determine convergence of sequences $\left\{x^{\alpha}_{oi},\ \sum_{j=1}^{N}{}' V^{\alpha}_{ji}\right\},\ \left\{\sum_{j=1}^{N+1}{}' V^{\alpha}_{ij}\right\}$, $\alpha = 1,2,\ \ldots$, because convergence of $V^{\alpha}_{i,N+1}$ cannot be inferred from compactness of the map sets $\mathbb{P}_i\left(x^{\alpha}_{oi},\ \sum_{j=1}^{N}{}' V^{\alpha}_{ji}\right)$ without free disposability of inputs, which cannot be applied when these map sets are compact. Therefore vectors $V_{i,N+1}$ of net output histories from an activity A_i , $(i = 1,2,\ \ldots,\ N)$, will be considered distinct from vectors of intermediate product output histories, by treating the activity correspondence as

$$\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon (L_\infty)_+^n \times (L_\infty)_+^m \to \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon\, 2^{(L_\infty)_+^p \times (L_\infty)_+^m}$$

$$(i = 1,2, \ldots, N) ,$$

where all net output histories are components of vectors in a net output space $(L_\infty)_+^p$, p a positive integer. For those activities which produce the complete products of the system which do not serve as intermediate products, this restructuring of the activity correspondences is no restriction. When intermediate product outputs of activities are sent to final output as spares, the alterations of the correspondences (10.1-1) are a minor restriction in that spares as final outputs must be treated as distinct from intermediate product outputs of the same kind and determined independently of the latter. Then the expression (10.1-6) becomes

$$\mathbb{P}(x) := \left\{u \;\varepsilon\; (L_\infty)_+^p : u \leqq \sum_{i=1}^{N} V_{i,N+1}, \left(V_{i,N+1}, \sum_{j=1}^{N}{}' V_{ij}\right) \varepsilon\; \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) ,\right.$$

(10.2-1)

$$\left. (i = 1,2, \ldots, N) , \sum_{i=1}^{N} x_{oi} \leqq x \right\} .$$

One additional assumption is made: that in this modified structure for the activity correspondences, the two vectors, $V_{i,N+1}$ and $\sum_{j=1}^{N}{}' V_{ij}$ are disposable by axiom $\mathbb{P}$.6S, i.e., each vector is weakly disposable independently of the other. This eminently reasonable assumption is the essential restriction involved in the restructuring described above.

It may be pointed out here, for emphasis, that the production model for network systems of producing activities is quite general. Cycles are not excluded, and alternative production activities in parallel are possible.

To continue with the consideration of axiom $\mathbb{P}.5$ for the production network, let $\{x^\alpha\}$ be an infinite sequence of vectors of exogenous input histories converging to the vector x^o, with $u^\alpha \varepsilon \mathbb{P}(x^\alpha)$ for all $\alpha = 1,2, \ldots$ and generating an infinite sequence $\{u^\alpha\}$ converging to u^o. Then, except for a finite number of elements of $\{x^\alpha\}$, the sequence $\{x^\alpha\}$ is bounded (in the relevant norm depending upon the topology) and accordingly under the same conditions a sequence $\left\{\sum_{i=1}^{N} x^\alpha_{oi}\right\}$ is bounded since $\sum_{i=1}^{N} x^\alpha_{oi} \leqq x^\alpha$ and the input rate time histories are nonnegative. Hence any allocation sequence $\left\{x^\alpha_{oi}\right\}$ is bounded in the relevant norm (excluding a finite number of them perhaps), $(i = 1,2, \ldots, N)$, and consequently there exist convergent subsequences $\left\{x^{\alpha_k}_{oi}\right\}$. For the subsequence $\{\alpha_k\}$, consider the sequences $\left\{\sum_{j=1}^{N}{}' V^{\alpha_k}_{ji}\right\}$ related to $\left\{u^{\alpha_k}\right\}$, $(i = 1,2, \ldots, N)$. These sequences are likewise bounded in the relevant norm due to property $\mathbb{P}.\mathbb{N}.1$. Consequently, there exists a subsequence $\{\alpha_\ell\} \subset \{\alpha_k\}$ such that

$$\left\{\sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ji}\right\} \to v^o_i \, , \left\{x^{\alpha_\ell}_{oi}\right\} \to x^o_{oi}$$

for $(i = 1,2, \ldots, N)$. Now $u^{\alpha_\ell} \varepsilon \mathbb{P}\left(x^{\alpha_\ell}\right)$ implies

$$\left(V^{\alpha_\ell}_{i,N+1}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ij}\right) \in \mathbb{P}_i\left(x^{\alpha_\ell}_{oi}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ji}\right), \ (i = 1,2, \ldots, N) ,$$

with

$$u^{\alpha_\ell} \leqq \sum_{i=1}^{N} V^{\alpha_\ell}_{i,N+1} .$$

Since $V^{\alpha_\ell}_{i,N+1}$ and $\sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ij}$ are independently weak disposable (i.e., the correspondence of A_i satisfies $\mathbb{P}$.6S in the vector pair $\left(V^{\alpha_\ell}_{i,N+1}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ij}\right)$)

$$V^{\alpha_\ell}_{i,N+1} \in \mathbb{P}_i\left(x^{\alpha_\ell}_{oi}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ji}\right), \ (i = 1,2, \ldots, N) .$$

Accordingly

$$\left\{V^{\alpha_\ell}_{i,N+1}\right\} \in \bigcup_{\{\alpha_\ell\}} \mathbb{P}_i\left(x^{\alpha_\ell}_{oi}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ji}\right) \quad (i = 1,2, \ldots, N) .$$

Since $\left\{\left(x^{\alpha_\ell}_{oi}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ji}\right)\right\}$ is a convergent sequence, and hence compact, it follows by Proposition (2.1.1-1) that

$$\bigcup_{\{\alpha^\ell\}} \mathbb{P}_i\left(x^{\alpha_\ell}_{oi}, \sum_{j=1}^{N}{}' V^{\alpha_\ell}_{ji}\right)$$

is compact. Hence there exists a subsequence $\{\alpha_\nu\} \subset \{\alpha_\ell\}$ such that

$$\left\{V^{\alpha_\nu}_{i,N+1}\right\} \to V^{o}_{i,N+1} , \ (i = 1,2, \ldots, N)$$

and

$$u^{o} \leqq \sum_{i=1}^{N} V^{o}_{i,N+1} .$$

Therefore the network correspondence $x \to \mathbb{P}(x)$ is closed, if the correspondences of the activities A_i obey $\mathbb{P}$.2S, $\mathbb{P}$.5, $\mathbb{P}$.5 Bis under the norm topology, or obey $\mathbb{P}$.2, $\mathbb{P}$.5, $\mathbb{P}$.5 Bis under a weak* topology, and vectors of net output time histories are distinct from vectors of intermediate time histories with each independently weakly disposable. Nothing in the foregoing arguments needs to be essentially altered if exogenous input rate time histories obey (10.1-8). However, the closure of the graph of the network correspondence cannot be obtained if inventories are permitted, since

$$\mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \text{ compact} \not\Rightarrow \hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N} V_{ji}\right)\right) \text{ compact.}$$

Turning now to the output disposal property for a network correspondence, depending upon whether the network correspondence is defined by (10.1-6) or modified to incorporate (10.1-7), independently of the assumptions made for the activity correspondences in this respect, property $\mathbb{P}$.6SS or $\mathbb{P}$.6S or $\mathbb{P}$.6 may apply.

For the production network output correspondence (10.1-5), (10.1-6) and the alterations as to disposal of inputs and outputs and use of inventories, there are two axioms on time spans of outputs which have to be considered, i.e., $\mathbb{P}$.T.1 and $\mathbb{P}$.T.2.

If the axiom $\mathbb{P}$.T.1 on initiation of output holds (see Section 2.1.3) for each activity correspondence (10.1-1), it will certainly hold for (10.1-6) or any alterations for disposal of outputs and inputs or for use of inventories. Similarly the axiom $\mathbb{P}$.T.2 on time extension of outputs holds for the network correspondence when the same applies to the correspondences (10.1-1), since $\bar{t}_{V_{i,N+1}} \leqq \bar{t}_{x_{oi}}$ for $i = 1,2, \ldots, N$ (See Section 2.1.3).

The axioms $\mathbb{L}$.T.1 and $\mathbb{L}$.T.2 on time extension of inputs and those on boundedness of efficient input histories require consideration of the network correspondence inverse to the output correspondence, and this matter is taken up in the following sections.

10.3 General Form of the Production Network Input Correspondence

Conforming to the definition (10.1-6) for the map sets of the network output correspondence, the network input correspondence, inverse to the output correspondence, is defined by

$$u \in (L_\infty)^m_+ \to \mathbb{L}(u) \in 2^{(L_\infty)^n_+} \tag{10.3-1}$$

$$\begin{aligned}\mathbb{L}(u) = \Bigg\{ x \in (L_\infty)^n_+ : x \geqq \sum_{i=1}^{N} x_{oi}, \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \in \mathbb{L}_i\left(\sum_{j=1}^{N+1}{}' V_{ij}\right), \\ (i = 1,2, \ldots, N), \sum_{i=1}^{N} V_{i,N+1} \geqq u \Bigg\}\end{aligned} \tag{10.3-2}$$

where

$$\left(\sum_{j=1}^{N+1} V_{ij}\right) \in (L_\infty)^m_+ \to \mathbb{L}_i\left(\sum_{j=1}^{N+1} V_{ij}\right), \quad (i = 1,2, \ldots, N)$$

are the activity input correspondences inverse to those of (10.1-1). In this representation inputs and outputs have been taken freely disposable. If histories of output rates for the production network are only strongly disposable, i.e., only $\mathbb{P}.6S \iff \mathbb{L}.6S$ holds, the last constraint in (10.3-2) is altered to

$$(10.3\text{-}3) \quad \sum_{i=1}^{N} (V_{i,N+1})_k = \theta_k u_k \ , \ \theta_k \ \varepsilon \ [1,+\infty) \ , \ (k = 1,2, \ldots, m)$$

and in the case of weak disposability of network outputs, i.e., only $\mathbb{P}.6$ applies, set $\theta_k = \theta \ \varepsilon \ [1,+\infty)$, $(k = 1,2, \ldots, m)$.

Input rate histories have been expressed as freely disposable in expression (10.3-2). If they are to be only strongly disposable, i.e., only $\mathbb{P}.3S \iff \mathbb{L}.3S$ applies, the first constraint of (10.3-2) is altered to:

$$(10.3\text{-}4) \quad x_j = \lambda_j \sum_{i=1}^{N} (x_{oi})_j \ , \ \lambda_j \ \varepsilon \ [1,+\infty) \ , \ (j = 1,2, \ldots, n) \ ,$$

and, in case input rate histories are only weakly disposable, one takes $\lambda_j = \lambda \ \varepsilon \ [1,+\infty)$, $(j = 1,2, \ldots, n)$.

The assumptions concerning the disposability of input and output rate histories for the production network can be made independently of the like assumptions for the correspondences of the activities of the network.

In the case where inventories of intermediate products and some exogenous inputs are allowed, some additional notation is required. Cumulative histories $(\hat{V}_{ij})_k$ of output rates produced by A_i for input to A_j , $(k = 1,2, \ldots, m)$, and cumulative histories of input rates

$(\hat{x}_{oi})_j$, $(j = 1,2, \ldots, n)$ are defined by (10.1-9) and (10.1-10) respectively. The cumulative transfers of intermediate products to A_i are defined by

$$(10.3\text{-}5) \qquad \hat{v}_i := \sum_{j=1}^{N}{}' \hat{V}_{ji}$$

and cumulative net output rate histories are defined by

$$(10.3\text{-}6) \qquad (\hat{u}(t))_k := \int_0^t (u(\tau))_k d\nu_k(\tau) \ , \ t \in [0,+\infty) \ , \ (k = 1,2, \ldots, m) \ .$$

For inventories in use of intermediate products and exogenous inputs

$$(10.3\text{-}7) \qquad \hat{\mathbb{L}}_i\left(\mathbb{L}_i\left(\sum_{j=1}^{N+1} V_{ij}\right)\right) := \left\{(\hat{x}_{oi},\hat{v}_i) : (x_{oi},v_i) \in \mathbb{L}_i\left(\sum_{j=1}^{N+1} V_{ij}\right)\right\}$$

defines the cumulative input histories to attain at least $\sum_{j=1}^{N+1} V_{ij}$. Then, for inventories, the map sets (10.3-2) for the input correspondence (10.3-1) inverse to (10.1-5), (10.1-6) may be expressed

$$\mathbb{L}(u) = \Bigg\{ x \in (L_\infty)_+^n : x_{\sigma_j} \geqq \left(\sum_{i=1}^{N} x_{oi}\right)_{\sigma_j} , (j = 1,2, \ldots, n) \ , \ \sum_{i=1}^{N} \hat{V}_{i,N+1} \geqq \hat{u} \ ,$$

$$(10.3\text{-}8) \qquad \left((\hat{x}_{oi})_{\sigma_1}, (\hat{x}_{oi})_{\sigma_2}, \ldots, (\hat{x}_{oi})_{\sigma_S}, (x_{oi})_{\sigma_{S+1}}, \ldots, (x_{oi})_{\sigma_n} , \sum_{j=1}^{N}{}' \hat{V}_{ji}\right) \in$$

$$\hat{\mathbb{L}}_i\left(\mathbb{L}_i\left(\sum_{j=1}^{N+1}{}' V_{ij}\right)\right) \ , \ (i = 1,2, \ldots, N) \Bigg\} \ .$$

See (10.1-13) for definition of $(x_{oi})_{\sigma_j}$.

That (10.3-8) is the inverse correspondence to (10.1-15) may be seen as follows:

$$\left(u \leqq \sum_{i=1}^{N} v_{i,N+1}\right) \Longleftrightarrow \left(\sum_{i=1}^{N} \hat{v}_{i,N+1} \geqq \hat{u}\right) ;$$

$$\left(\sum_{i=1}^{N} (\hat{x}_{oi})_{\sigma_j} \leqq (\hat{x})_{\sigma_j}\right) \Longleftrightarrow \left(x_{\sigma_j} \geqq \sum_{i=1}^{N} (x_{oi})_{\sigma_j}\right), \; (j = 1,2, \ldots, S) ;$$

$$\left(\sum_{j=1}^{N+1}{}' \hat{v}_{ij}\right) \varepsilon \; \hat{\mathbb{P}}_i\left(\mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' v_{ji}\right)\right) \Longleftrightarrow \sum_{i=1}^{N+1}{}' v_{ij} \; \varepsilon \; \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' v_{ji}\right)$$

$$\Longleftrightarrow \left(x_{oi}, \sum_{j=1}^{N}{}' v_{ji}\right) \varepsilon \; \mathbb{L}_i\left(\sum_{i=1}^{N+1}{}' v_{ij}\right) \Longleftrightarrow$$

$$\left((\hat{x}_{oi})_{\sigma_1}, (\hat{x}_{oi})_{\sigma_2}, \ldots, (\hat{x}_{oi})_{\sigma_S}, (x_{oi})_{\sigma_{S+1}}, \ldots, (x_{oi})_{\sigma_n}, \sum_{j=1}^{N}{}' v_{ji}\right) \varepsilon$$

$$\hat{\mathbb{L}}_i\left(\mathbb{L}_i\left(\sum_{j=1}^{N+1}{}' v_{ij}\right)\right), \; (i = 1,2, \ldots, N) .$$

These implications arise because the time rate histories involved are nonnegative, and are compared pointwise in time.

10.4 Properties of the Production Network Input Correspondence

It follows from the definition of the production network input correspondence as the inverse correspondence of the production network output correspondence, that properties $\mathbb{L}.1$, $\mathbb{L}.2$, $\mathbb{L}.3$ (or $\mathbb{L}.3S$ or $\mathbb{L}.3SS$), $\mathbb{L}.4.2$, $\mathbb{L}.5$, $\mathbb{L}.6$ (or $\mathbb{L}.6S$ or $\mathbb{L}.6SS$) hold for the network input correspondence as equivalent properties to those shown in Section 10.2 for the output correspondence, with property $\mathbb{L}.5$ holding under the additional assumptions made for property $\mathbb{P}.5$.

It remains to consider axioms $\mathbb{L}.T.1$ and $\mathbb{L}.T.2$ on time extension of inputs. For this purpose two additional assumptions need to be considered for the network, which do not arise when intermediate products are not explicitly considered for the production system as in Chapter 2.

In order to produce a vector of summable output rate histories, i.e., bounded amounts of each kind of output, it should be possible to do this with vectors of summable input rate histories of intermediate products. In the case where the output histories have bounded supports, one would certainly accept this as a general principle. As an axiom this property is extended to cases where the output histories are summable. If a single activity produced all net outputs and each activity received its intermediate products from only one activity, the need for the axiom would be moot, because axiom $\mathbb{L}.T.1$ holding for this activity would imply that output could be produced with vectors of summable input rate histories of intermediate products, which in turn requires only summable input rate histories for all activities obeying $\mathbb{L}.T.1$. However, under the generality of definition of the inverse correspondence given by (10.3-2) and the alterations following, the axiom needs to be introduced.

Next, concerning bounded supports for net output rate histories, when all outputs cease it is to be expected that no further production of intermediate products is required. However, under the generality of the definition of the network inverse (input) correspondence, this property for the activities does not imply the same for the network, and needs to be taken as a network axiom.

Hence, the following two network axioms are introduced:

$\mathbb{P}.\mathbb{N}.2$ If $u \in (L_\infty)_+^m$ is summable on $[0,+\infty)$ in each component as a vector of net output rate histories for a production network, there exist x and V_{ij}, $(i,j = 1,2, \ldots, N)$, of exogenous and intermediate product input rate histories, with summable components, which yield u.

$\mathbb{P}.\mathbb{N}.3$ If $\bar{t}_u < \infty$ for a production network, and x yields u with intermediate product transfers V_{ij}, $(i,j = 1,2, \ldots, N)$,

$$\left.\begin{array}{l} y_i(t) = x_i(t)\ ,\ t \in [0,\bar{t}_u] \\ y_i(t) = 0\ ,\ t \in (\bar{t}_u +\infty) \end{array}\right. \quad (i = 1,2, \ldots, n)$$

$$\left.\begin{array}{l} (W_{ij}(t))_k = (V_{ij}(t))_k\ ,\ t \in [0,\bar{t}_u] \\ (W_{ij}(t))_k = 0\ ,\ t \in (\bar{t}_u,+\infty) \end{array}\right. \quad \begin{array}{l} (k = 1,2, \ldots, m) \\ (i = 1,2, \ldots, N) \\ (j = 1,2, \ldots, N) \end{array}$$

also yields u.

With these two axioms it is clear that properties $\mathbb{L}.\mathbb{T}.1$ and $\mathbb{L}.\mathbb{T}.2$ hold for the production network input correspondence when the same properties hold for the activity input correspondences.

Finally, concerning the axioms on efficient subsets (see Section 2.2.4), the notion of efficient vectors x of input rate histories as introduced for an abstract production correspondence (see Section 2.2.3) needs to be strengthened somewhat for a production network. Referring to (10.3.2) for the production network input set $\mathbb{L}(u)$, and defining efficiency solely with respect to exogenous inputs, $x \in \mathbb{E}(u)$ if and only if

$$\text{(a)} \quad x = \sum_{i=1}^{N} x_{oi} \; , \; \sum_{i=1}^{N} V_{i,N+1} = u$$

$$\text{(b)} \quad \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon \; \mathbb{L}_i\left(\sum_{j=1}^{N+1}{}' V_{ij}\right), \; (i = 1,2, \ldots, N)$$

$$\text{(c)} \quad \left(y_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \notin \; \mathbb{L}_i\left(\sum_{j=1}^{N+1}{}' V_{ij}\right) \quad \text{for} \quad y_{oi} \lneq x_{oi} \; , \; (i = 1,2, \ldots, N) \; .$$

These conditions may be satisfied for any intermediate product transfers $\left(\sum_{j=1}^{N}{}' V_{ji}\right)$, $(i = 1,2, \ldots, N)$ such that (b) and (c) are satisfied. The production network is defined in such a general way that the conditions (a), (b), (c) do not imply

$$\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon \; \mathbb{E}_i\left(\sum_{j=1}^{N+1}{}' V_{ij}\right) .$$

In effect, inefficient intermediate product production is possible for $x \; \varepsilon \; \mathbb{E}(u)$ under this definition. By specializing the production network one might exclude such possibilities. The choice made here is to retain the generality of the definition of a production network and strengthen the definition of efficiency for a network by replacing (b) and (c) with

$$\text{(d)} \quad \left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \varepsilon \; \mathbb{E}_i\left(\sum_{j=1}^{N+1}{}' V_{ij}\right), \; (i = 1,2, \ldots, N) \; .$$

On this basis, any one of the axioms of Section 2.2.4 on boundedness or total boundedness for the efficient subsets of the input correspondences of the activities carries over to the production network, because x is a finite sum of vectors each of which belong to a bounded (or totally bounded) efficient subset.

By these developments (Sections 10.1, 10.2, 10.3 and 10.4) the production correspondences of a general production network are defined and exhibit the same properties (axioms) as developed earlier for a more abstract production structure which does not explicitly display intermediate products. In the following section the production network will be made more specific by introducing technical coefficients.

10.5 Technical Coefficient Form of a Network Production Correspondence

Up to now each activity of the production network has been treated as an abstract process. For purposes of application the correspondences of the activities need to be expressed in terms of technical data.

The operation of the activity A_i is taken to be characterized by an intensity function $z_i \varepsilon (L_\infty)_+$, defined by $z_i(t)$ for $t \varepsilon [0,+\infty)$, $i = (1,2, \ldots, N)$. This function expresses at each time t the rate at which A_i is operating. The physical unit of $z_i(t)$ may refer to some good or service per unit time, or merely be the reciprocal of time. By axiom $\mathbb{P}$.T.1 production cannot be instantaneous. There are certain natural bounds on the intensity functions z_i depending upon the process involved. Let

$$(10.5\text{-}1) \qquad \bar{z} : = (\bar{z}_1, \ldots, \bar{z}_N) \;\varepsilon\; (L_\infty)_+^N$$

denote these bounds. Inputs of goods or services into an activity and outputs of goods or services are driven by these intensity functions in terms of technical coefficient functions.

For exogenous inputs let

$$(10.5\text{-}2) \qquad \mathbb{A}^{(i)} := \| a_{i1} a_{i2} \cdots a_{in} \| \ , \ (i = 1,2, \ldots, N)$$

denote row matrices of coefficient functions $a_{ij}(t)$, $t \in [0,+\infty)$, $(j = 1,2, \ldots, n)$ of $(L_\infty)_+$, which express at each time t the number of units of the j^{th} exogenous input required per unit intensity of the i^{th} network activity. The coefficients $a_{ij}(t)$ are taken dependent upon time so that "learning" or any other gradual change in technical relationship may be allowed. Even abrupt technical change, which can be programmed in advance, may be expressed by the time varying technical coefficients.

Assumptions made concerning the coefficient functions $\mathbb{A}^{(i)}$ are:

(I) For each $i \in \{1,2, \ldots, N\}$, and $t \in [0,+\infty)$ except for a subset of measure zero, there exists $j \in \{1,2, \ldots, n\}$ such that $a_{ij}(t) > 0$. SOME EXOGENOUS INPUT IS REQUIRED FOR EACH ACTIVITY (see $\mathbb{P}.\mathbb{N}.1$).

(II) For each $j \in \{1,2, \ldots, n\}$ there exists $i \in \{1,2, \ldots, N\}$ such that $a_{ij}(t) > 0$ on a subset of $[0,+\infty)$ of positive measure. EACH EXOGENOUS INPUT IS REQUIRED.

Normally, if a coefficient function a_{ij} is positive on a subset of $[0,+\infty)$ of positive measure it is positive almost everywhere on $[0,+\infty)$, except for abrupt technical change. Assumptions (I) and (II) are worded to allow for the possibility of abrupt technical change where one newly introduced input replaces another.

The outputs of the activities are vectors of time histories in $(L_\infty)^m_+$, driven by the intensity functions z_i , $(i = 1,2, \ldots, N)$, $z = (z_1, z_2, \ldots, z_N) \in (L_\infty)^N_+$, in terms of output technical coefficient

functions

$$(10.5\text{-}3) \qquad \mathbb{C}^{(i)} := ||c_{i1} c_{i2} \cdots c_{im}|| , \ (i = 1,2, \ldots, N)$$

where $c_{ij} \in (L_\infty)_+$, $(j = 1,2, \ldots, m)$ and $c_{ij}(t)$ for $t \in [0,+\infty)$ expresses at each time t the amount of the j^{th} net output produced per unit intensity of the i^{th} network activity.

The assumptions made concerning the output coefficient functions are:

(III) For each $i \in \{1,2, \ldots, N\}$, there exists $j \in \{1,2, \ldots, m\}$ such that $c_{ij}(t) > 0$ on a subset of $[0,+\infty)$ of positive measure. EACH ACTIVITY CAN PRODUCE SOME OUTPUT.

(IV) For each $j \in \{1,2, \ldots, m\}$, there exists $i \in \{1,2, \ldots, N\}$ such that $c_{ij}(t) > 0$ on a subset of $[0,+\infty)$ of positive measure. EACH OUTPUT IS PRODUCIBLE.

The output histories of the activities serve both net output and intermediate products. The technical coefficient functions for intermediate product inputs are

$$(10.5\text{-}4) \qquad \bar{\mathbb{A}}^{(i)} := ||\bar{a}_{i1} \bar{a}_{i2} \cdots \bar{a}_{im}|| ,$$

where $\bar{a}_{ij} \in (L_\infty)_+$, and $\bar{a}_{ij}(t)$, $t \in [0,+\infty)$ expresses at the time t the number of units of the j^{th} kind of output $(j = 1,2, \ldots, m)$ required per unit intensity of the i^{th} network activity $(i = 1,2, \ldots, N)$. Non-negativity of the functions $\bar{a}_{ij}$ is all that is required, since an activity may produce only intermediate or only final product, as well as both.

As notation let

$$\mathbb{A} := (\mathbb{A}^{(1)}\mathbb{A}^{(2)} \cdots \mathbb{A}^{(N)})^T$$

(10.5-5) $$\bar{\mathbb{A}} := (\bar{\mathbb{A}}^{(1)}\mathbb{A}^{(2)} \cdots \mathbb{A}^{(N)})^T$$

$$\mathbb{C} := (\mathbb{C}^{(1)}\mathbb{C}^{(2)} \cdots \mathbb{C}^{(N)})^T .$$

Then, the output correspondence for the production network has output sets $\mathbb{P}(x)$, under free disposability of inputs and outputs, given by:

(10.5-6) $$\mathbb{P}(x) = \Big\{u \in (L_\infty)^m_+ : 0 \leqq z \leqq \bar{z} \quad z\mathbb{A} \leqq x \ , \ z(\mathbb{C} - \bar{\mathbb{A}}) \leqq 0 \ , \ u \leqq z(\mathbb{C} - \bar{\mathbb{A}})\Big\} \ , \quad x \in (L_\infty)^n_+ .$$

If inputs are not freely disposable the second constraint of (10.5-6) is changed to

(10.5-7) $$(z\mathbb{A})_j = \frac{1}{\lambda_j} x_j \ , \ \lambda_j \in [1,+\infty) \ , \ (j = 1,2, \ldots, n) \ ,$$

where $(z\mathbb{A})_j$ denotes the j^{th} component of the vector $(z\mathbb{A})$. Similarly, if outputs are not freely disposable, the last constraint of (10.5-6) is changed to

(10.5-8) $$u_j = \theta_j(z(\mathbb{C} - \bar{\mathbb{A}}))_j \ , \ \theta_j \in [0,1] \ , \ (j = 1,2, \ldots, m) \ .$$

The expression for $\mathbb{P}(x)$ states that the vectors of output histories u obtainable from the vector x of input histories are those generated by vectors z of intensity functions for the activities such that the histories of exogenous inputs required for the entire network do not exceed those available and all intermediate product histories required by these intensity histories of activity operation can be produced.

In effect the third constraint sorts out the "mix" of vectors z such that intermediate product requirements can be met.

Inversely, the network input correspondence map sets $\mathbb{L}(u)$ are given (under free disposal of inputs and outputs) by

$$\mathbb{L}(u) = \left\{ x \in (L_\infty)_+^n : 0 \leqq z \leqq \bar{z} \ , \ z(\mathbb{C} - \bar{\mathbb{A}}) \geqq u \right.$$

(10.5-9)

$$\left. x \geqq z\mathbb{A} \right\} , \ u \in (L_\infty)_+^m \ .$$

The alterations when free disposal of inputs and outputs are not permitted can be made as shown by (10.5-7) and (10.5-8).

In the case where inventories are permitted the constraints of (10.5-6) become

$$0 \leqq z \leqq \bar{z} \ ; \ \sum_{i=1}^{N} \left(\int_0^t z_i(\tau) a_{i\sigma_j}(\tau) d\mu_{\sigma_j}(\tau) \right) \leqq \int_0^t (x(\tau))_{\sigma_j} d\mu_{\sigma_j}(\tau) \quad \text{for}$$

$$t \in [0,+\infty) \ , \ (j = 1,2, \ldots, S) \ ; \ \sum_{i=1}^{N} z_i a_{i\sigma_j} \leqq (x)_{\sigma_j}$$

(10.5-10)

$$\text{for} \quad (j = (S+1, S+2, \ldots, n)) \ ;$$

$$\sum_{i=1}^{N} \left(\int_0^t z_i(\tau)(c_{ij}(\tau) - \bar{a}_{ij}(\tau)) d\nu_j(\tau) \right) \geqq 0 \ , \ t \in [0,+\infty) \ , \ (j = 1,2, \ldots, m) \ ,$$

$$u \leqq z(\mathbb{C} - \bar{\mathbb{A}}) \ .$$

When inventories are permitted for the inverse (input correspondence), the inequalities of (10.5-9) become

$$0 \leqq z \leqq \bar{z} \ , \ \sum_{i=1}^{N} \left(\int_0^t z_i(\tau)(c_{ij}(\tau) - \bar{a}_{ij}(\tau)) \right) d\nu_j(\tau) \geqq u_j(t) \ , \ t \in [0,+\infty) \ ,$$

$$(j = 1,2, \ldots, m) \ , \ x \geqq z\mathbb{A} \ .$$

When inputs and outputs are not freely disposable, the constraints are changed further by use of (10.5-7) and (10.5-8).

Computation for maximal value of output or minimal value of cost, for (10.5-6) and (10.5-9) respectively, is carried out by representing the functions involved only at discrete points of time. Clearly the mathematical programs related thereto can be too large for calculation. In any practical case, the special structure of the production network can facilitate computation. Such valuations are not alone what is of interest. The typical problem in production planning is one of smooth loading resources. For these reasons, and to illustrate the evolutionary character of dynamic production correspondences, certain special structures will be considered in the following sections.

10.6 Special Structures

Cycles in transfers of intermediate products will be excluded for the special structures to be considered, because they introduce additional complicating detail for the dynamic generation of the network production correspondence. With this exclusion the activities A_i $(i = 1,2, \ldots, N)$ may be ordered so that all intermediate product inputs required by an activity A_i can be obtained from the activities $A_1, A_2, \ldots, A_{i-1}$, if any are needed $(i = 2,3, \ldots, N)$.

10.6.1 Output Correspondence for a Production Network with Given Facilities and Single Output Activities

In shipbuilding and other construction projects, as well as many fine structured production systems, the output of the network may be one or more final products with each activity of the network yielding a single product, serving as intermediate product only, final product only or both intermediate and final product. The activities need not be unique, that is some may provide alternative production activities for the same output.

For some convenient unit of time, consider a time grid $t = 0,1,2, \ldots$. Use the following conventions:

> Inputs (exogenous and intermediate product) are applied at uniform rates during each period $[(t-1),t)$, $(t = 1,2,3, \ldots)$ and charged at time $(t-1)$ for the period $[(t-1),t)$.
>
> Outputs of production during $[(t-1),t)$ are forthcoming at time t and transferred as intermediate product at that time, $(t = 1,2,3, \ldots)$.
>
> The intensity function z_i for A_i $(i = 1,2, \ldots, N)$, is constant during each period $[(t-1),t)$ and this value is used in conjunction with technical coefficients at time $(t-1)$, $(t = 1,2,3, \ldots)$.

When each activity yields only a single output, the output technical coefficient functions simplify to

$$(10.6.1\text{-}1) \quad ¢^{(i)} := \|00 \cdots 0C_i 00 \cdots 0\| \ , \ i \in \{1, \ldots, m\} \ .$$

The production network is constrained by the capacities of certain plant and equipment. The maximal input rates of the services of these facilities are limited by their capacities. The activities using these services have thereby a limitation on intensity of operation. If each plant and equipment service is peculiar to a single activity, the given facilities imply in conjunction with the natural bounds (10.5-1) certain bounds on the intensity functions. In this case these bounds are denoted by

$$(10.6.1\text{-}2) \qquad \bar{\bar{z}} := (\bar{\bar{z}}_1, \bar{\bar{z}}_2, \ldots, \bar{\bar{z}}_N) \in (L_\infty)^N_+ .$$

In case the service of a given plant or equipment is shared by two or more activities, there results a bound on the sum of such activity intensities. This case will be discussed at the end of this section. Then omitting the services of the given facilities as exogenous inputs, the correspondence for the production network with given facilities relates the vector x of unconstrained exogenous input histories to the subsets $\mathbb{P}(x)$ of vectors of output histories.

With the activities ordered as described above, the technical coefficients for intermediate products take the form

$$(10.6.1\text{-}3) \quad \bar{\mathbb{A}}^{(i)} := \| \bar{a}_{i1} \bar{a}_{i2} \cdots \bar{a}_{i,i-1} 0 \cdots 0 \| \qquad (i = 1,2, \ldots, N) .$$

Concerning the net outputs of the system, let

$$(10.6.1\text{-}4) \quad \delta_{N+1,j}(t) \qquad (t = 1,2,3, \ldots) , \; (j = 1,2, \ldots, N)$$

denote the fraction of the output rate history of A_j going to net output at the time t . These coefficients are nonnegative, equal to

or less than unity, with $\delta_{N+1,j}(t)$ positive for at least one $j \in \{1,2, \ldots, N\}$ for a subset of $[0,+\infty)$ of positive measure. The coefficients (10.6.1-4) represent a preallocation of activity outputs when they supply both intermediate and net product. Then the time history of net output of the system is expressed by

$$(10.6.1\text{-}5) \quad V_{N+1}(t) := \sum_{j=1}^{N} \delta_{N+1,j}(t) z_j(t-1) \mathbb{C}^{(j)}(t) \ , \ (t = 1,2,3, \ldots) \ .$$

Since the intermediate product outputs of an activity may supply one or more of the successor activities in the network, let $\Delta_{ji}(t)$, $(t = 1,2,3, \ldots)$ be nonnegative real numbers denoting a preallocation of j^{th} activity output at time t to the activity A_i, $(j = 1,2, \ldots, (N-1))$, $(i = (j+1),(j+2), \ldots, N)$ as intermediate product inputs. These coefficients satisfy

$$(10.6.1\text{-}6) \quad \Delta_{ji}(t) \geqq 0 \ , \ \sum_{i=j+1}^{N} \Delta_{ji}(t) = 1 - \delta_{N+1,j}(t) \ , \ (t = 1,2,3, \ldots) \ .$$

The intensity functions possible for the correspondences

$$\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \to \mathbb{P}_i\left(x_{oi}, \sum_{j=1}^{N}{}' V_{ji}\right) \quad (i = 1,2, \ldots, N)$$

are then constrained for each $i \in \{1,2, \ldots, N\}$ by the following system of inequalities in which accumulations of intermediate product inventories are allowed.

(1) $z_i(t) \geqq 0$, $(t = 0,1,2,3, \ldots)$

(2) $z_i(t)a_{ij}(t) \leqq (x_{oi}(t))_j$, $(j = 1,2, \ldots, n)$, $(t = 0,1,2,3, \ldots)$

(3) $z_i(0)\bar{a}_{ij}(0) \leqq 0$

$$z_i(t)\bar{a}_{ij}(t) + \sum_{\tau=1}^{t-1} z_i(\tau)\bar{a}_{ij}(\tau) \leqq \sum_{\tau=1}^{t} V_{ji}(\tau) \qquad \begin{array}{l}(j = 1,2, \ldots, (i-1)) \\ (t = 1,2,3, \ldots)\end{array}$$

(4) $z_i(t) \leqq \bar{\bar{z}}_i(t)$, $(t = 0,1,2,3, \ldots)$.

For the expression of these constraints a preallocation $x_{oi}(t)$, $(t = 0,1,2,3, \ldots)$, $(i = 1,2, \ldots, N)$ of the system exogenous input history $x \in (L_\infty)^n_+$ has been made subject to $\sum_{i=1}^{N} x_{oi}(t) \leqq x_o(t)$, $(t = 0,1,2,3, \ldots)$. The full possibilities of production result from the union of the sets of network output histories obtainable from all preallocations of x_o and preallocations of outputs as intermediate products and net outputs.

The constraints (1) and (4) merely bound the intensity functions by zero and the maximal values determined by the natural bounds and the limitations due to the given facilities. The constraints (2) reflect a preallocation of exogenous input histories to the activities and require during each unit of time that inputs by the activities do not exceed these allocations. Constraints (3) allow inventories of intermediate products and require that the current input during any unit of time to an activity does not exceed the amount stored for that activity.

In order to make a specific calculation of net output histories for the production network, the intensity functions at any time $(t = 0,1,2,3, \ldots)$ will be taken at the maximal values possible subject to the constraints (1), ..., (4) and the preallocations made of exogenous inputs, intermediate products and net outputs. In this way a so-called "Greedy" solution will

be constructed. Clearly this particular solution is only one possibility among a very large number. But the greedy solution serves the purpose of being an initial solution which may be altered by time substitution of input histories to yield other solutions, among which those with smooth loading of resources are of particular interest.

A calculation of the greedy output time histories for the production network requires an evolutionary determination, successively, of the intensity functions $z_i(t)$, $(t = 0,1,2,3, \ldots)$ in the order $(i = 1,2, \ldots, N)$. In making this calculation the preallocations $\delta_{N+1,j}(t)$, $\Delta_{ji}(t)$, $(t = 1,2,3, \ldots)$ of the outputs of an activity are used.

Since the activity A_1 precedes all others, no intermediate product input to A_1 is possible. Accordingly the intensity history $z_1(t)$, $(t = 0,1,2,3, \ldots)$ for the greedy solution is determined solely from the system constraints (1), (2) and (4). The inequalities (1), (2) and (4) imply

$$0 \leqq z_1(t) \leqq \bar{\bar{z}}_1(t) \; ; \; z_1(t) \leqq \frac{x_{01}(t)_j}{a_{1j}(t)} \qquad (j = 1,2, \ldots, n) .$$

Accordingly, the greedy solution for the history $z_1(t)$, $(t = 0,1,2, \ldots)$ is

$$(10.6.1\text{-}7) \qquad z_1(t) = \operatorname{Min} \left[R^1(t) \; , \; \bar{\bar{z}}_1(t) \right] , \; (t = 0,1,2,3, \ldots)$$

where

$$(10.6.1\text{-}8) \quad R^i(t) := \operatorname*{Min}_{j \varepsilon S^i(t)} \left\{ \frac{(x_{oi}(t))_j}{a_{ij}(t)} \right\} , \; (i = 1,2, \ldots, N) ,$$

and

$$(10.6.1\text{-}9)\quad S^i(t) := \{j \in \{1,2,\ldots,n\} : a_{ij}(t) > 0\}\ ,\ (i = 1,2,\ldots,N)\ .$$

Due to assumptions (a) for $\mathbb{A} := (\mathbb{A}^{(1)}\mathbb{A}^{(2)} \cdots \mathbb{A}^{(N)})^T$, $S^i(t)$ is not empty for $(t = 0,1,2,3,\ldots)$. If the exogenous input coefficient functions a_{ij} vary only because of learning effects the set $S^i(t)$ does not depend upon t .

Next concerning activity A_2 , it may possibly (but not necessarily) need the output of A_1 as intermediate product input. To allow for this, the system constraint (3) must be used as well as (2). For this purpose, note that

$$(10.6.1\text{-}10)\quad \begin{array}{c} V_{ji}(t) = \Delta_{ji}(t)c_j(t)z_j(t-1)\ ,\ (t = 1,2,3,\ldots)\ , \\ j \in \{1,2,\ldots,N\}\ ,\ j \neq i\ , \end{array}$$

and the constraint system implies

$$z_2(t)\bar{a}_{21}(t) \leqq \left(\sum_{\tau=1}^{t} \Delta_{12}(\tau)c_1(\tau)z_1(\tau-1) - \sum_{\tau=1}^{t-1} z_2(\tau)\bar{a}_{21}(\tau)\right)\ .$$

If $\bar{a}_{21}(t) = 0$ for some time $t = 1,2,3,\ldots$ this constraint is not effective. Hence, we need only be concerned when $\bar{a}_{21}(t) > 0$. For this purpose, and to handle the general case, define

$$(10.6.1\text{-}11)\quad \Sigma^i(t) = \{j \in \{1,2,\ldots,(i-1)\} : \bar{a}_{ij}(t) > 0\}\ ,\ (i = 1,2,\ldots,N)\ .$$

Then the constraint system (3) requires

$$z_2(t) \leqq N^2(t) := \begin{cases} \sum_{\tau=1}^{t} \Delta_{12}(\tau)c_1(\tau)z_1(\tau-1) - \sum_{\tau=0}^{t-1} z_2(\tau)\bar{a}_{21}(\tau) , & \Sigma^2(t) \neq \emptyset \\ \infty \quad \text{for} \quad \Sigma^2(t) = \emptyset \quad (t = 1,2,3, \ldots) . \end{cases}$$

Note that from (3) and (1) that $z_i(0)\bar{a}_{ij}(0) = 0$, $(i = 1,2, \ldots, N)$. Together the system constraints (2) and (3) require for $(t = 1,2,3, \ldots)$, that

$$z_2(t) \leqq W^2(t) := \mathrm{Min}\left\{R^2(t), \bar{\bar{z}}_2(t), N^2(t)\right\} , \quad (t = 1,2,3, \ldots)$$

$$z_2(0) \leqq W^2(0) := \begin{cases} \mathrm{Min}\left[R^2(0), \bar{\bar{z}}_2(0)\right] & \text{for} \quad \Sigma^2(0) = \emptyset \\ 0 \quad \text{for} \quad \Sigma^2(0) \neq \emptyset . \end{cases}$$

For the greedy solution: $z_2(t) = W^2(t)$, $(t = 0,1,2,3, \ldots)$. The general greedy solution is obtained from:

(10.6.1-12) $z_i(t) = W^i(t)$, $(t = 0,1,2,3, \ldots)$, $(i = 1,2, \ldots, N)$

where

(10.6.1-13) $$W^i(0) := \begin{cases} \mathrm{Min}\left(R^i(0), \bar{\bar{z}}_i(0)\right) & \text{for} \quad \Sigma^i(0) = \emptyset \\ 0 \quad \text{for} \quad \Sigma^i(0) \neq \emptyset , \end{cases}$$

(10.6.1-14) $W^i(t) := \mathrm{Min}\left(R^i(t), \bar{\bar{z}}_i(t), N^i(t)\right)$, $(t = 1,2,3, \ldots)$

$$(10.6.1\text{-}15)\quad N^i(t) := \begin{cases} \underset{j\varepsilon\Sigma^i(t)}{\text{Min}} \left[\dfrac{\sum_{\tau=1}^{t} \Delta_{ji}(\tau)c_j(\tau)W^{(j)}(\tau-1) - \sum_{\tau=0}^{t-1} W^i(\tau)\bar{a}_{ij}(\tau)}{\bar{a}_{ij}(t)} \right] & \text{for } \Sigma^i(t) \neq \emptyset \\ +\infty \quad \text{for } \Sigma^i(t) = \emptyset & (t = 1,2,3, \ldots) . \end{cases}$$

The evolutionary character of the solution is evident. In the order $(i = 1,2,3, \ldots)$ one calculates $W^i(t)$ for $(t = 0,1,2,3, \ldots)$. For each $i \varepsilon \{1,2, \ldots, N\}$, the value $W^j(t)$, $(j = 1,2, \ldots, (i-1))$ needed for the calculation of $W^i(t)$ have been previously determined. The final greedy output history is then given by

$$(10.6.1\text{-}16)\quad V_{N+1}(t) = \sum_{j=1}^{N} \delta_{N+1,j}(t) \cdot W^j(t-1)\cancel{C}^{(j)}(t) , \quad (t = 1,2,3, \ldots) .$$

The full complexity of the output histories possibly obtained from a vector x of histories of exogenous inputs is only realized when, in addition to the alternatives for preallocationg exogenous inputs histories to the activities and the alternatives for preallocating transfers of activity outputs as intermediate products and net outputs, one considers also the possibilities for obtaining the same output histories by operating the activities with intensities less than the maximal values (10.6.1-2), as well as different output histories.

Ordinarily the preallocations of exogenous input flows and intermediate product transfer flows, in conjunction with the intensity bounds $\bar{\bar{z}} = (\bar{\bar{z}}_1, \bar{\bar{z}}_2, \ldots, \bar{\bar{z}}_N)$, will be imperfectly balanced so that the intensity

functions for some activities may be operated at less than maximal intensity and still yield the output streams of the greedy solution. Such shifts in the intensity functions and the related alterations of input histories used represent Time Substitutions of the input histories applied. The possibility of intermediate product inventories supports time substitution.

In case inventories of certain exogenous inputs are possible, the inequality constraints (2) for the network are modified to:

$$\text{(2a)} \quad \sum_{\tau=0}^{t} z_i(\tau) a_{i\nu_j}(\tau) \leqq (\hat{x}_{oi}(t))_{\nu_j} \quad , \; (j = 1,2, \ldots, S)$$

$$\text{(2b)} \quad z_i(t) a_{i\nu_j}(t) \leqq (x_{oi}(t))_{\nu_j} \quad , \; (j = S+1, S+2, \ldots, n) \; .$$

Then the greedy solutions for the intensity functions take the form

$$z_i(t) = W^i(t) \; , \; t = 0,1,2,3, \ldots$$

$$W^i(0) := \begin{cases} \text{Min}\left[R^i(0), \bar{\bar{z}}_i(0), \bar{R}^i(0)\right] & \text{for} \quad \Sigma^i(0) = \emptyset \\ 0 \quad \text{for} \quad \Sigma^i(0) \neq \emptyset & \end{cases}$$

$$W^i(t) := \text{Min}\left[R^i(t), \bar{R}^i(t), \bar{\bar{z}}_i(t), N^i(t)\right] \; , \; (t = 1,2,3, \ldots)$$

$$S^i(t) := \left\{ j : j \in \{S+1, S+2, \ldots, n\} \; , \; a_{i\nu_j}(t) > 0 \right\} \; , \; (i = 1,2, \ldots, N)$$

$$\bar{S}^i(t) := \left\{ j : j \in \{1,2, \ldots, S\} \; , \; a_{i\nu_j}(t) > 0 \right\} \; , \; (i = 1,2, \ldots, N)$$

$$\bar{R}^i(0) := \begin{cases} \underset{j \in \bar{S}^i(0)}{\text{Min}} \left[\dfrac{(x_{oi}(0))_{\nu_j}}{a_{i\nu_j}(0)} \right] & \text{for} \quad \bar{S}^i(0) \neq \emptyset \\ +\infty \quad \text{for} \quad \bar{S}^i(0) = \emptyset & \end{cases}$$

$$\bar{R}^i(t) := \begin{cases} \underset{j\varepsilon\bar{S}^i(t)}{\text{Min}} \left[\dfrac{(\hat{x}_{oi}(t))_{\nu_j} - \sum_{\tau=0}^{t-1} W^i(\tau)\bar{a}_{i\nu_j}(\tau)}{a_{i\nu_j}(t)} \right] & \text{for } \bar{S}^i(t) \neq 0 \\ +\infty \quad \text{for } \bar{S}^i(t) = \emptyset \quad (t = 1,2,3, \ldots) \end{cases}$$

with $R^i(t)$ and $N^i(t)$ defined by (10.6.1-8) and (10.6.1-15) respectively.

The inventories permitted for intermediate products and exogenous inputs may also be bounded in amount. Detail of this kind can be added to the greedy solution.

Initially it was assumed that each plant and equipment service was peculiar to a single activity. Suppose now that there are "f" fixed resources, with input coefficients for the activities defined by

(10.6.1-17) $\tilde{A}^{(i)} := ||\tilde{a}_{i1}\tilde{a}_{i2} \cdots \tilde{a}_{if}||$, $(i = 1,2, \ldots, N)$

where $\tilde{a}_{ij} \varepsilon (L_\infty)_+$ with $\tilde{a}_{ij}(t)$, $t \varepsilon [0,+\infty)$, $(j = 1,2, \ldots, f)$ denotes at time t the amount of the services of the j^{th} fixed facility required per unit intensity of the i^{th} network activity. Then, if $(d_1,d_2, \ldots, d_f) \varepsilon (L_\infty)^f_+$ represents the time histories of the services of these fixed facilities available per unit time, it is required that the intensity histories $z = (z_1,z_2, \ldots, z_N) \varepsilon (L_\infty)^N_+$ satisfy

(10.6.1-18) $\sum_{i=1}^{N} \tilde{a}_{ij}(\sigma)z_i(\sigma) \leqq d_j(\sigma)$, $(j = 1,2, \ldots, f)$, $(\sigma = 0,1,2,3, \ldots)$.

The shared capacity bounds so permitted are vectors from the set

$$Z := \Big\{ z \in (L_\infty)_+^N : z_i(t) = z_i(\sigma)\ ,\ t \in [\sigma, \sigma+1)\ ,\ i = (1,2,\ldots,N)\ ,$$

$$\text{(10.6.1-19)} \qquad \sum_{i=1}^{N} \tilde{a}_{ij}(\sigma) z_i(\sigma) \leqq d_j(\sigma)\ ,\ (j = 1,2,\ \ldots,\ f)$$

$$z_i(\sigma) \leqq \bar{z}_i(\sigma)\ ,\ (i = 1,2,\ldots,N)\ ,\ (\sigma = 0,1,2,3,\ \ldots) \Big\} .$$

Then the constraints (1), ..., (4) for $(i = 1,2, \ldots, N)$ must be taken in conjunction with $\overset{=}{z} \in Z$, and the entire output set $\mathbb{P}(x)$ for the network results from the union of all possible preallocations of exogenous inputs, intermediate products and net outputs taken with all possible $\overset{=}{z} \in Z$ and the satisfaction of the inequalities (1), ..., (4), where inequality signs are permitted to apply.

10.6.2 Input Correspondence for a Production Network with Given Facilities and Activities with Single Output

It is convenient to express the output histories required as cumulative requirements. As defined earlier in Section 10.1, let $\hat{V}_i$ denote the cumulative output history of A_i. A finite output horizon T will be taken for the definition of the input correspondence of the network. Let

$$\text{(10.6.2-1)} \qquad \sigma := (T - t)\ ,\ (t = T, T-1, T-2, \ldots)$$

denote the number of time units preceding T.

As in the treatment of the output correspondence we shall assume that there is more than one net output of the system, say $p > 1$. Initially, each activity will be taken to yield a single output which may be only either intermediate product or final product. The ordering of the activities may be made so that activities A_i $(i = N, N-1, \ldots, N-p+1)$ yield these final products, retaining the property that activity A_i

can supply only A_j for $j = (i+1), (i+2), \ldots, N$, $i = 1, 2, \ldots, (N-1)$. In this framework, the network correspondence

$$(V_N, V_{N-1}, \ldots, V_{N-p+1}) \to \mathbb{L}(V_N, V_{N-1}, \ldots, V_{N-p+1})$$

will be studied, calculating backward from the horizon T , i.e., forward with $(\sigma = 0, 1, 2, \ldots)$.

As a first approach it will be assumed that none of the activities produce the same product, i.e., there are no alternative processes in the production network. This simplifies the calculation, because then one need not preassign how much of a needed product will be forthcoming from each alternative activity. Also it will be assumed that a pre-allocation of the services of fixed capital equipment is reflected in the intensity function bounds (10.6.1-2), when two or more activities share the same capital service.

The constraints for determining the greedy solution are:

$$\text{(a)} \quad 0 \leqq z_i(\sigma) \leqq \bar{\bar{z}}_i(\sigma) \text{ , } (i = 1, 2, \ldots, N) \text{ , } \sigma = 0, 1, 2, 3, \ldots$$

$$\text{(b)} \quad \begin{cases} c_i(0) z_i(1) \leqq \hat{V}_i(0) - \hat{V}_i(1) \text{ , } (i = N, N-1, \ldots, (N-p+1)) \\ c_i(\sigma-1) z_i(\sigma) + \sum\limits_{\tau=1}^{\sigma-1} c_i(\tau-1) z_i(\tau) \leqq \hat{V}_i(0) - \hat{V}_i(\sigma) \text{ ,} \\ \qquad (i = N, N-1, \ldots, N-p+1) \text{ , } (\sigma = 2, 3, 4, \ldots) \end{cases}$$

$$\text{(c)} \quad \begin{cases} c_i(1) z_i(2) \leqq \sum\limits_{j=(i+1)}^{N} \bar{a}_{ji}(1) z_j(1) \text{ , } (i = 1, 2, \ldots, (N-p)) \\ c_i(\sigma-1) z_i(\sigma) + \sum\limits_{\tau=2}^{\sigma-1} c_i(\tau-1) z_i(\tau) \leqq \sum\limits_{\tau=1}^{\sigma-1} \sum\limits_{j=(i+1)}^{N} \bar{a}_{ji}(\tau) z_j(\tau) \\ \qquad (i = 1, 2, \ldots, (N-p)) \text{ , } (\sigma = 3, 4, 5, \ldots) \end{cases}$$

(d) $(x_{oi}(\sigma))_j \geqq a_{ij}(\sigma)z_i(\sigma)$, $(j = 1,2, \ldots, n)$; $(\sigma = 1,2,3, \ldots)$

(e) $\hat{V}_{ji}(\sigma) \geqq \sum_{\tau=\sigma}^{\infty} \bar{a}_{ij}(\tau)z_i(\tau)$, $(j = 1,2, \ldots, (i-1))$, $(\sigma = 1,2,3, \ldots)$.

The constraints (a) merely bound the intensity functions by zero and their maximal values as determined by the given facilities. The constraints (b) require in cumulative terms that the output histories for the final products of the network do not exceed the histories $V_N, V_{N-1}, \ldots, V_{N-p+1}$ sought for these products. The intensity functions permitted by the constraints (b) need not satisfy the equality signs for all periods $(\sigma = 1,2,3, \ldots)$, since storage of final product is permitted. By extending the number of periods used, the cumulative requirements can be met by storing earlier produced final product. All such possibilities are time substitutable intensity histories.

The constraints (c) require in cumulative terms that the output histories of nonfinal product producing activities do not exceed the requirements for the outputs of these activities. As in the cases of the constraints (b), the equality signs of (c) need not apply for all periods by storing earlier production.

Constraints (d) and (e) merely drive exogenous input histories non-cumulatively and intermediate product transfer histories cumulatively, under an assumption of free disposal of exogenous input histories and inventory storage of intermediate products. For the calculation of vectors $x \in (L_\infty)^n_+$ belonging to $\mathbb{L}(V_N, V_{N-1}, \ldots, V_{N-p+1})$, (e) will not actually be used, since it only involves internal bookkeeping.

A greedy solution for the system (a), ..., (d) is obtained by finding the maximal values possible for the intensity functions at all periods ($\sigma = 1,2, \ldots$) . During each period, the intensity for each activity is taken as the largest value possible. In this way the shortest total time will be taken to produce the output histories $V_N, V_{N-1}, \ldots, V_{N-p+1}$.

To construct the greedy solution, consider the first set of constraints of (b) concerning the production of final products during the period $\sigma \in [1,0)$ or equivalently $t \in [T-1,T]$. When taken with the constraints (a), it is implied that

$$z_i(1) \leqq E^i(1) := \operatorname{Min}\left[\frac{\hat{V}_i(0) - \hat{V}_i(1)}{c_i(0)}, \bar{\bar{z}}_i(1)\right], (i = N, N-1, \ldots, N-p+1) .$$

For the greedy solution: $z_i(1) = E^i(1)$, $(i = N, N-1, \ldots, N-p+1)$.

Next consider the second constraint set of (b). When taken with the constraints (a) and the greedy solution is used for $z_i(1)$, $(i = N, N-1, \ldots, N-p+1)$ one obtains

$$z_i(\sigma) \leqq E^i(\sigma) := \operatorname{Min}\left[\frac{\hat{V}_i(0) - \hat{V}_i(\sigma) - \sum_{\tau=1}^{\sigma-1} c_i(\tau-1) z_i(\tau)}{c_i(\sigma-1)}, \bar{\bar{z}}_i(\sigma)\right]$$

$$(i = N, N-1, \ldots, N-p+1) , (\sigma = 2,3, \ldots) .$$

The greedy solutions are those for which $z_i(\sigma) = E^i(\sigma)$, $(\sigma = 2,3, \ldots)$, calculated successively.

Having calculated the greedy solution intensities for all final product activities during all periods, we may proceed with the greedy solutions for the intermediate product producing activities. Using the first set of inequalities in (c) with (a), and the greedy solutions

for $z_i(\sigma)$, $(i = N, N-1, \ldots, (N-p+1))$, $(\sigma = 1,2,3, \ldots)$, it is implied that

$$z_i(2) \leqq E^i(2) := \operatorname{Min}\left[\frac{\sum_{j=(i+1)}^{N} \bar{a}_{ji}(1)E^j(1)}{c_i(1)} , \bar{\bar{z}}_i(2)\right] ,$$

$$(i = (N-p), (N-p-1), \ldots, 3,2,1) .$$

The greedy solution is $z_i(2) = E^i(2)$, $(i = 1,2, \ldots, (N-p))$. From the second set of constraints of (c), it is required for period 3 that

$$z_i(3) \leqq E^i(3) := \operatorname{Min}\left[\frac{\sum_{\tau=1}^{2} \sum_{j=(i+1)}^{N} \bar{a}_{ji}(\tau)E^j(\tau) - c_i(1)E^i(1)}{c_i(2)} , \bar{\bar{z}}_i(3)\right]$$

$$(i = (N-p) , (N-p-1), \ldots, 3,2,1) .$$

Here again the greedy solution is $z_i(3) = E^i(3)$ for $(i = (N-p), (N-p-1), \ldots, 3,2,1)$. For period 3, the functions $E^i(3)$ are calculated in descending order of activity number, starting with $i = (N-p)$ and using the solutions for $E^j(\sigma)$, $(\sigma = 1,2)$, $(j = N, N-1, \ldots, (N-p), (N-p-1), \ldots, (i+1))$ for calculating $E^i(3)$.

Hence the general form of the greedy solution is

(10.6.2-2) $z_i(\sigma) = E^i(\sigma)$, $(\sigma = 1,2, \ldots)$, $(i = 1,2, \ldots, N)$,

where

$$E^i(1) := \mathrm{Min}\left[\frac{\hat{V}_i(0) - \hat{V}_i(1)}{c_i(0)}\ ,\ \bar{\bar{z}}_i(1)\right] \tag{10.6.2-3}$$

$$(i = N, N-1, N-2,\ \ldots,\ N-p+1)\ .$$

$$E^i(\sigma) := \mathrm{Min}\left[\frac{\hat{V}_i(0) - \hat{V}_i(\sigma) - \sum_{\tau=1}^{\sigma-1} c_i(\tau-1)z_i(\tau)}{c_i(\tau-1)}\ ,\ \bar{\bar{z}}_i(\sigma)\right] \tag{10.6.2-4}$$

$$(\sigma = 2,3,4,\ \ldots)\ ,\ (i = N, N-1,\ \ldots,\ N-p+1)\ .$$

$$E^i(1) := 0\ ,\ (i = (N-p),(N-p-1),\ \ldots,\ 2,1)$$

$$E^i(2) := \mathrm{Min}\left[\frac{\sum_{j=(i+1)}^{N} \bar{a}_{ji}(1)E^j(1)}{c_i(1)}\ ,\ \bar{\bar{z}}_i(2)\right] \tag{10.6.2-5}$$

$$(i = (N-p),(N-p-1),\ \ldots,\ 2,1)\ .$$

$$E^i(\sigma) := \mathrm{Min}\left[\frac{\sum_{\tau=1}^{\sigma-1}\ \sum_{j=(i+1)}^{N} \bar{a}_{ji}(\tau)E^j(\tau) - \sum_{\tau=2}^{\sigma-1} c_i(\tau-1)E^i(\tau)}{c_i(\sigma-1)}\ ,\ \bar{\bar{z}}_i(\sigma)\right] \tag{10.6.2-6}$$

$$(i = (N-p),(N-p-1),\ \ldots,\ 2,1)\ ,\ (\sigma = 3,4,5,\ \ldots)\ .$$

The steps for constructing the greedy solution for the intensity histories are:

(i) For the cumulative values $\hat{V}_{N-\alpha}(\sigma)$, $(\sigma = 1,2,\ \ldots)$, $(\alpha = 0,1,2,\ \ldots,\ (p-1))$ of the final output histories $V_N(\sigma), V_{N-1}(\sigma),\ \ldots,\ V_{N-p+1}(\sigma)$, $(\sigma = 1,2,3,\ \ldots)$, calculate $z_i(\sigma) = E^i(\sigma)$, $(\sigma = 1,2,\ \ldots)$ from (10.6.2-2), (10.6.2-3) for $(i = N, N-1,\ \ldots,\ (N-p+1))$.

(ii) With the values obtained in (i), calculate $z_i(\sigma) = E^i(\sigma)$, $(\sigma = 1,2, \ldots)$ successively for $i = (N - p)$, $i = (N - p - 1)$, ... , $i = 2$, $i = 1$, in that order, using for $i = \alpha$, the solutions obtained for $E^i(\sigma)$, $(i = N, N - 1, N - 2, \ldots, (\alpha + 1))$, $(\sigma = 2,3, \ldots)$.

In making these calculations for $\sigma = 1,2,3, \ldots$, the process is terminated when no further positive amounts of the intensity values are needed.

The vector of input histories $x \in (L_\infty)^n_+$ corresponding to this greedy solution is obtained from (d) by taking the equal sign, i.e.,

$$(10.6\text{-}2\text{-}7) \quad x_o(\sigma) = [E^1(\sigma)E^2(\sigma) \cdots E^N(\sigma)]\mathbb{A}(\sigma) , \quad (\sigma = 0,1,2,3, \ldots)$$

where $\mathbb{A}(\sigma) = (\mathbb{A}^{(1)}(\sigma)\mathbb{A}^{(2)}(\sigma) \cdots \mathbb{A}^{(N)}(\sigma))^T$.

Under these greedy solution, individual vectors of histories of exogenous inputs for the activities of the network are:

$$(10.6.2\text{-}8) \quad \begin{aligned} x_{oi}(\sigma) &= \left(a_{i1}(\sigma)E^i(\sigma), a_{i2}(\sigma)E^i(\sigma), \ldots, a_{in}(\sigma)E^i(\sigma)\right) \\ &(i = 1,2, \ldots, N) , \quad (\sigma = 0,1,2, \ldots) . \end{aligned}$$

The cumulative transfers of intermediate products needed to support the i^{th} activity for the greedy solution are obtained from (e) by using the equality sign, i.e.,

$$(10.6.2\text{-}9) \quad \begin{aligned} \hat{V}_{ji}(\sigma) &= \sum_{\tau=\sigma}^{\infty} \bar{a}_{ij}(\tau)E^i(\tau) , \quad (j = 1,2, \ldots, (i - 1)) , \\ &(\sigma = 1,2, \ldots) , \quad (i = (N-p),(N-p-1), \ldots, 2,1) . \end{aligned}$$

The cumulative net output history required from an activity producing intermediate product is

$$(10.4.2\text{-}10)\qquad \hat{V}_i(\sigma) = \sum_{\tau=\sigma}^{\infty} \sum_{j=(i+1)}^{N} \bar{a}_{ji}(\tau)E^i(\tau) \ , \ (\sigma = 1,2,3, \ldots) \ ,$$
$$(i = (N-p),(N-p-1), \ldots, 2,1) \ ,$$

which is supplied by the cumulative output of A_i , given by

$$(10.4.2\text{-}11)\qquad \hat{V}_i(\sigma) = \sum_{\tau=\sigma-1}^{\infty} c_i(\tau+1)E^i(\tau) \ , \ (\sigma = 1,2,3, \ldots)$$
$$(i = (N-p),(N-p-1), \ldots, 2,1) \ .$$

This greedy solution is evidently highly evolutionary. Lags are not exogenous. The time delays between production events are endogenous, depending upon the intensity values used for the activities, illustrating that fixed time lags cannot rigorously be used for dynamic models of production.

The greedy solution is but one of very, very many possible intensity histories which may be used to attain the vector $(V_N, V_{N-1}, \ldots, V_{N-p+1}) \in (L_\infty)_+^p$, of output histories. Any solution obtained by not using the equal signs in the constraint system (a), ..., (d) is a feasible set of intensity histories for an input program of $\mathbb{L}(V_N, V_{N-1}, \ldots, V_{N-p})$. Each vector $x_o(\sigma)$, $(\sigma = 1,2,3, \ldots)$ determined by these feasible intensity histories is a time substitutable program of exogenous inputs to obtain the output histories $V_N, V_{N-1}, \ldots, V_{N-p+1}$ and they define the isoquant of the map set $\mathbb{L}(V_N, V_{N-1}, \ldots, V_{N-p+1})$. It should be noted here that a preallocation of the services of given facilities has been reflected in the intensity bounds (10.6.1-2) used for the greedy solution.

A new dimension is added for dynamic models in addition to the traditional factor substitution of the static theory of production, namely: TIME SUBSTITUTION for inputs. Each of the programs generated from the constraint system (a), ..., (d), provides a new distribution over time of the same total amounts of exogenous inputs, and is therefore a time substitution of the greedy solution. Even though all capital services come from fixed facilities, and the exogenous inputs are not substitutable as factors, there may be considerable time substitution possible for these exogenous input histories. Time substitution is the essential dynamic aspect of factor application. In a complex case, both factors and time substitution may be intermixed. Thus one can appreciate the full possibilities for the dynamic phenomena of production.

One important problem of production planning is that of constant loading the application of resources. The greedy solution is likely to provide an undesirably variable loading. If the time histories of the intensity functions were constant and the technical coefficients did not vary much with time, a smooth loading of exogenous inputs and intermediate product production would result. This suggests that one may seek to use the time substitution possibilities for the intensity function in order to modify them for constancy. The greedy solution provides the least total span of production time to obtain the output histories of final products. An activity A_i is critical during any period $[\sigma,\sigma - 1)$, $(\sigma = 1,2,3, \ldots)$ if $z_i(\sigma)$ may not be decreased without extending the total production time of the greedy solution. When an activity A_i is not critical during a period $[\sigma,\sigma + 1)$, the greedy solution may be altered by reducing the capacity bound $\bar{\bar{z}}_i(\sigma)$ until A_i becomes critical during $[\sigma,\sigma + 1)$. The altered greedy

solution obtained by decreasing $\bar{\bar{z}}_i(\sigma)$ is a time substitution for $z_i(\sigma)$, $(\sigma = 1,2,3, \ldots)$ which does not extend the total production time needed to obtain $(V_N, V_{N-1}, \ldots, V_{N-p+1}) \in (L_\infty)_+^p$. By all of the time substitutions of this type one may alter the greedy solution toward smoother loading of resources. If all activities became critical for all periods by these alterations, the resulting loading of resources would be the smoothest that can be obtained without increasing the total production time required to obtain $(V_N, V_{N-1}, \ldots, V_{N-p+1})$. Further smoothing may be sought by *time substitutions which extend the total time of the* project.

Initially, it has been assumed for the input correspondence that when activities share the services of certain fixed resources, a pre-allocation of this sharing is made and reflected in the intensity bounds for such activities, and (10.4.2-12) is a definition of the input correspondence subject to this restriction. This qualification may be removed as follows:

Suppose there are f fixed resources, with input coefficients for the activities defined as in (10.6.1-17) by

$$(10.6.2\text{-}12) \quad \tilde{\mathbb{A}}^{(i)} := \|\tilde{a}_{i1}\tilde{a}_{i2} \cdots \tilde{a}_{if}\|, \ (i = 1,2, \ldots, N)$$

where $\tilde{a}_{ij} \in (L_\infty)_+$, with $\tilde{a}_{ij}(t)$, $t \in [0,+\infty)$, $(j = 1,2, \ldots, f)$ denotes at each time t the amount of the j^{th} fixed resource required per unit intensity of the i^{th} network activity. Then, if $(d_1, d_2, \ldots, d_f) \in (L_\infty)_+^f$ represents the time histories of the number of units of these resources available per unit time, it is required that the intensity histories $z = (z_1, z_2, \ldots, z_N) \in (L_\infty)_+^N$ satisfy

$$\sum_{i=1}^{N} \tilde{a}_{ij}(\sigma) z_i(\sigma) \leqq d_j(\sigma)$$

$$(i = 1,2, \ldots, f) , (\sigma = 1,2,3, \ldots) .$$

Then, the shared capacity bounds permissable are defined by

$$\mathfrak{z} : = \left\{ z \in (L_\infty)_+^N : z_i(t) = z_i(\sigma) , t \in [\sigma, \sigma - 1) , (i = 1,2, \ldots, N) , \right.$$

$$(10.6.2\text{-}13) \qquad \sum_{i=1}^{N} \tilde{a}_{ij}(\sigma) z_i(\sigma) \leqq d_{ij}(\sigma) , j = 1,2, \ldots, f$$

$$\left. z_i(\sigma) \leqq \bar{z}_i(\sigma) , (i = 1,2, \ldots, N) , (\sigma = 1,2,3, \ldots) \right\} .$$

Then one may seek by optimization or heuristic the shortest time span for the greedy solution with the constraints (a), ..., (d) supplemented by $\bar{\bar{z}} \in \mathfrak{z}$.

Clearly there are many dynamically variable possibilities in sharing resources, giving rise to problems of optimal allocation of shared resources. Heuristics for this kind of problem can be developed. Here we have a time substitution in sharing resources.

To extend the model of the network input correspondence further, suppose that it is possible for two intermediate product producing activities to produce the same output, say A_α , $A_{\alpha+1}$, for $1 \leqq \alpha < (N - p)$. Then the constraints (c) are altered for $i = \alpha$ and $i = \alpha + 1$ to state

$$(10.6.2\text{-}14) \quad \sum_{i=\alpha}^{\alpha+1} \left(c_i(\sigma - 1) z_i(\sigma) + \sum_{\tau=2}^{\sigma-1} c_i(\tau - 1) z_i(\tau) \right) \leqq \sum_{\tau=1}^{\sigma-1} \sum_{j=(\alpha+2)}^{N} \bar{a}_{j,\alpha+1}(\tau) z_j(\tau) .$$

Then, without preallocation of the outputs of A_α and $A_{\alpha+1}$ to supply A_j $(j = \alpha + 2, \alpha + 3, \ldots, N)$, the greedy solution is indeterminant. Hence, in order to accommodate the role of such substitutable activities some kind of policy rule is needed. Ordinarily, the duplicating processes (activities) can be ordered by their efficiency, say $(\alpha + 1)$ is more efficient or more preferred for whatever reason applies. Then, one may replace (10.6.2-14) by

$$c_{\alpha+1}(\sigma-1)z_{\alpha+1}(\sigma) + \sum_{\tau=2}^{\sigma-1} c_{\alpha+1}(\tau-1)z_{\alpha+1}(\tau) \leqq \sum_{\tau=1}^{\sigma-1} \sum_{j=\alpha+2}^{N} \bar{a}_{j,\alpha+1}(\tau)z_j(\tau),$$

(10.6.2-15)
$$\left(c_\alpha(\sigma-1)z_\alpha(\sigma) + \sum_{\tau=2}^{\sigma-1} c_\alpha(\tau-1)z_\alpha(\tau)\right) \leqq$$

$$\left(\sum_{\tau=1}^{\sigma-1} \sum_{j=\alpha+2}^{N} \bar{a}_{j,\alpha+1}(\tau)z_j(\tau) - c_{\alpha+1}(\sigma-1)\cdot z_{\alpha+1}(\sigma) - \sum_{\tau=2}^{\sigma-1} c_{\alpha+1}(\tau-1)z_{\alpha+1}(\tau)\right).$$

Note that the coefficients $\bar{a}_{j,\alpha+1}(\sigma)$, $\bar{a}_{j,\alpha}(\sigma)$ have to have the same values. Moreover, if the two activities are duplicates the same modification applies, because it does not matter which is used.

If two final output activities yield the same output, a preallocation of the contribution of each to the required output of that type is required. Obviously the system can function if the least preferred of the two is left idle, but then the total span of production time may be increased. By comparing and evaluating costs of the greedy solutions for various preallocations one may cost out the saving in time. The problem faced here is a complex problem of optimization.

The simplification made by assuming that each activity output could be either intermediate product or final product, but not both, may be relaxed as follows:

Suppose there are $p > 1$ net outputs. Supplement the network with activities $A_{N+1}, A_{N+2}, \ldots, A_{N+p}$, which merely collect these net outputs. Denote the intensities of operation of these nodes during the interval $[\sigma, \sigma - 1)$ by

$$(10.6.2\text{-}16) \qquad z_{N+j}(\sigma) \ , \ (\sigma = 1,2,3, \ldots) \ , \ (j = 1,2, \ldots, p)$$

constrained only to be nonnegative. The output coefficients of these activities are

$$(10.6.2\text{-}17) \qquad c_{N+j}(\sigma) \ , \ (j = 1,2, \ldots, p) \ , \ (\sigma = 0,1,2,3, \ldots)$$

expressing the amount of the j^{th} net output collected per unit intensity of the activity A_{N+j} during $[\sigma + 1, \sigma)$. The intensities (10.6.2-16) may be understood so that

$$(10.6.2\text{-}18) \qquad c_{N+j}(\sigma) = 1 \ , \ (j = 1,2, \ldots, p) \ , \ (\sigma = 0,1,2,3, \ldots) \ .$$

Concerning inputs to the activities A_{N+j}, $(j = 1,2, \ldots, p)$ only outputs of the activities A_i $(i = 1,2, \ldots, N)$ will be needed. Coefficients expressing these requirements are:

$$(10.6.2\text{-}19) \qquad \bar{a}_{N+j,i}(\sigma) = 1 \ , \ (\sigma = 0,1,2, \ldots) \ , (j = 1,2, \ldots, p) \ , \ (i = 1,2, \ldots, N)$$

expressing the amount of the output of the i^{th} activity required per unit intensity of the collecting activity A_{N+j}.

With these additions the constraints for the greedy solution for the net output histories $V_j(\sigma)$, $(j = 1,2, \ldots, p)$, $(\sigma = 0,1,2, \ldots)$ become

$$\text{(a)}\quad 0 \leqq z_i(\sigma) \leqq \overset{=}{z}_i(\sigma)\ ,\ (i = 1,2,\ \ldots,\ N)\ ,\ z_{N+j}(\sigma) \geqq 0\ ,$$
$$(j = 1,2,\ \ldots,\ p)\ ,\ (\sigma = 0,1,2,3,\ \ldots)$$

$$\text{(b)}\quad \begin{cases} c_{N+j}(0)z_{N+j}(1) \leqq \hat{V}_j(0) - \hat{V}_j(1) \quad (j = 1,2,\ \ldots,\ p) \\ c_{N+j}(\sigma-1)z_{N+j}(\sigma) + \sum\limits_{\tau=1}^{\sigma-1} c_{N+j}(\tau-1)z_{N+j}(\tau) \leqq \hat{V}_j(0) - \hat{V}_j(\sigma) \\ \qquad (j = 1,2,\ \ldots,\ p)\ ,\ (\sigma = 2,3,4,\ \ldots) \end{cases}$$

$$\text{(c)}\quad \begin{cases} c_i(1)z_i(2) \leqq \sum\limits_{j=(N+1)}^{N+p} \bar{a}_{ji}(1)z_j(1) \quad (i = 1,2,\ \ldots,\ N) \\ c_i(\sigma-1)z_i(\sigma) + \sum\limits_{\tau=2}^{\sigma-1} c_i(\tau-1)z_i(\tau) \leqq \sum\limits_{\tau=1}^{\sigma-1} \sum\limits_{j=(i+1)}^{N+p} \bar{a}_{ji}(\tau)z_j(\tau) \\ \qquad (i = 1,2,\ \ldots,\ N)\ ,\ (\sigma = 3,4,5,\ \ldots) \end{cases}$$

$$\text{(d)}\quad (x_{oi}(\sigma))_j \geqq a_{ij}(\sigma)z_i(\sigma)\ ,\ (j = 1,2,\ \ldots,\ n)\ ,\ (i = 1,2,\ \ldots,\ N)$$

$$\text{(e)}\quad \hat{V}_{ji}(\sigma) \geqq \sum_{\tau=\sigma}^{\infty} \bar{a}_{ij}(\tau)z_i(\tau)\ ,\ (j = 1,2,\ \ldots,\ \min\,(N,i-1))\ ,$$
$$(i = 1,2,\ \ldots,\ (N+p))\ .$$

Under these constraints the greedy solution for the intensity functions $z_i(\sigma)$, $i = 1,2,\ \ldots,\ N+p$, $\sigma = 0,1,2,\ \ldots$ are

$$(10.6.2\text{-}20)\quad z_i(\sigma) = E^i(\sigma)\ ,\ (\sigma = 1,2,3,\ \ldots)\ ,\ (i = 1,2,\ \ldots,\ N+p)$$

where

$$(10.6.2\text{-}21)\qquad E^{N+j}(1) := (\hat{V}_j(0) - \hat{V}_j(1))\ ,\ (j = 1,2,\ \ldots,\ p)\ .$$

$$(10.6.2\text{-}22)\qquad \begin{gathered} E^{N+j}(\sigma) := \hat{V}_j(0) - \hat{V}_j(\sigma) - \sum_{\tau=1}^{\sigma-1} E^{N+j}(\tau) \\ (j = 1,2,\ \ldots p)\ ,\ (\sigma = 2,3,\ \ldots)\ . \end{gathered}$$

$$(10.6.2\text{-}23)\qquad E^i(1) := 0 \;,\; (i = 1,2, \ldots, N) \;.$$

$$(10.6.2\text{-}24)\qquad E^i(2) := \operatorname{Min}\left[\frac{\sum_{j=(i+1)}^{N+p} \bar{a}_{ji}(1)E^j(1)}{c_i(1)} \;,\; \bar{\bar{z}}_i(2)\right] \quad (i = 1,2, \ldots, N) \;.$$

$$(10.6.2\text{-}25)\qquad E^i(\sigma) := \operatorname{Min}\left[\frac{\sum_{\tau=1}^{\sigma-1}\sum_{j=i+1}^{N+p} \bar{a}_{ji}(\tau)E^j(\tau) - \sum_{\tau=2}^{\sigma-1} c_i(\tau-1)E^i(\tau)}{c_i(\sigma-1)} \;,\; \bar{\bar{z}}_i(\sigma)\right]$$

$$(i = 1,2, \ldots, N) \;,\; (\sigma = 3,4,5, \ldots) \;.$$

10.6.3 Job Shop Production

Consider a shop of N machine tools performing operations for metal fabrication such as milling, grinding, drilling, etc. These tools function as activities $A_1, A_2, \ldots, A_N$ for the job shop production. The job shop fabricates a variety of different products in lots of varying size. Assume that production planning is to be done for $p > 1$ products. These products do not necessarily require the use of the same machines, and the initial and final machining operation need not be the same. However, it is assumed that the orderings of activities used in manufacture by the products are coherent in that there exists an ordering of the activities of the shop say $A_1, A_2, \ldots, A_N$ such that in the routing of each product processing by an activity A_i does not require previous processing by A_j $(j = i + 1, i + 2, \ldots, N)$, $(i = 1,2,3,4, \ldots, N)$.

It is also assumed for this model of production that there are no duplicate machine tools, i.e., each activity performs a distinct operation for the job shop. In effect the activities $A_1, \ldots, A_N$ are machine tool centers, with duplicate machines located in one center.

Let $\hat{V}_j(t)$, $j = 1,2, \ldots, p$, denote cumulative *output* schedules for the p products. An output horizon T is chosen to coincide with the latest time at which any of these products is due, with time units counted backward by $(\sigma = 0,1,2,3, \ldots)$ as defined by (10.6.2-1).

The policy for processing these products will be taken as: products with the latest final due date will be processed first in reverse time $(\sigma = 1,2,3, \ldots)$. Hence the products are ordered in descending index starting with p , according to latest final due date, from the latest to the earliest. Note that if σ_j is the latest due date of the j^{th} product

$$(10.6.3\text{-}1) \quad \hat{V}_j(\sigma) = \hat{V}_j(\sigma_j) \quad \text{for} \quad \sigma \leqq \sigma_j \qquad (j = 1,2, \ldots, p) \; .$$

As at the end of Section 10.6.2, introduce activities A_{N+j} $(j = 1,2, \ldots, p)$ with intensities $z_{N+j}(\sigma)$, $(\sigma = 1,2,3, \ldots)$, to collect the outputs of the p products, with technical coefficient functions $c_{N+j}(\sigma)$, $\bar{a}_{N+j,i}(\sigma)$ as defined by (10.6.2-17), (10.6.2-18) and (10.6.2-19).

Each of the work centers $A_1, A_2, \ldots, A_N$ has a certain capacity for intensity of operation, which need not be constant in time because of routine maintenance. The histories of these capacities represent bounds upon the intensities of operating the activities denoted by

$$(10.6.3\text{-}2) \qquad \bar{\bar{z}} := (\bar{\bar{z}}_1, \bar{\bar{z}}_2, \ldots, \bar{\bar{z}}_N) \in (L_\infty)^N_+ \; .$$

Since each activity A_i may potentially operate on any product, the intensity of operation of A_i on product j during $[\sigma, \sigma - 1)$, equivalent to $[T - \sigma, T - \sigma + 1)$, is denoted by

(10.6.3-3) $z_i^j(\sigma)$, $(j = 1,2, \ldots, p)$, $(i = 1,2, \ldots, N)$, $(\sigma = 1,2,3, \ldots)$.

Concerning the output technical coefficient functions for A_i , the amount of the j^{th} product processed per unit intensity of the i^{th} activity applied to the j^{th} product during $[\sigma + 1, \sigma)$ is denoted by

(10.6.3-4) $c_i^j(\sigma)$, $(j = 1,2, \ldots, p)$, $(i = 1,2, \ldots, N)$, $(\sigma = 0,1,2,3, \ldots)$.

Concerning inputs to the machine centers, exogenous inputs, in addition to the services of the machine centers already accounted for by the intensity bounds, can be neglected for practical purposes, especially if the machine tools are automated as a system. For the illustration discussed here, only inputs of work in process will be considered. Let

(10.6.3-5) $\bar{a}_{i\alpha}^j(\sigma)$, $(j = 1,2, \ldots, p)$, $(i = 1,2, \ldots, N)$, $\alpha = 1,2, \ldots, (i-1)$

denote the amount of finished work in process required from A_α on product j per unit intensity of operating A_i during $[\sigma, \sigma - 1)$.

Then, under the policy that products with the latest final due date will be processed first in reverse time σ , the constraints to greedy solution are:

$$(10.6.3\text{-}6) \quad (a) \begin{cases} z_i^j(\sigma) \geqq 0 \ , \ z_i^p(\sigma) \leqq \bar{\bar{z}}_i(\sigma) \ , \ (j = p, p-1, \ldots, 2,1) \ , \\ \qquad (i = 1,2, \ldots, N) \\ z_i^j(\sigma) \leqq \bar{\bar{z}}_i(\sigma) - \sum\limits_{\alpha=j+1}^{p} z_i^\alpha(\sigma) \ , \ (j = p-1, p-2, \ldots, 2,1) \ , \\ \qquad (i = 1,2, \ldots, N) \end{cases}$$

$$(10.6.3\text{-}7)\quad (b)\begin{cases} c_{N+j}(0)z_{N+j}(1) \leqq \hat{V}_j(0) - \hat{V}_j(1)\ , \quad (j = p, p-1, \ldots, 2, 1) \\ c_{N+j}(\sigma-1)z_{N+j}(\sigma) + \sum\limits_{\tau=1}^{\sigma-1} c_{N+j}(\tau-1)z_{N+j}(\tau) \leqq \hat{V}_j(0) - \hat{V}_j(\sigma) \\ \qquad (\sigma = 2,3,4, \ldots) \end{cases}$$

$$(10.6.3\text{-}8)\quad (c)\begin{cases} c_i^j(1)z_i^j(2) \leqq \sum\limits_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^j(1)z_\alpha^j(1) \\ c_i^j(\sigma-1)z_i^j(\sigma) + \sum\limits_{\tau=2}^{\sigma-1} c_i^j(\tau-1)z_i^j(\tau) \leqq \sum\limits_{\tau=1}^{\sigma-1} \sum\limits_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^j(\tau)z_\alpha^j(\tau) \\ \qquad (j = p, p-1, \ldots, 2, 1)\ , \quad (i = 1, 2, \ldots, N) \\ \qquad (\sigma = 3,4,5, \ldots)\ . \end{cases}$$

For the greedy solution of the intensity functions of this system, the intensities for outputs are given by

$$(10.6.3\text{-}9)\quad \begin{aligned} z_{N+j}(1) &= E^{N+j}(1)\ , \quad (j = p, p-1, \ldots, 2, 1) \\ z_{N+j}(\sigma) &= E^{N+j}(\sigma)\ , \quad (\sigma = 2,3,4, \ldots) \end{aligned}$$

where $E^{N+j}(1)$ and $E^{N+j}(\sigma)$ are defined by (10.6.2-21) and (10.6.2-22). Next, consider activity A_i $(i = 1,2, \ldots, N)$ for products $(p, p-1, \ldots, 2,1)$ in this order. Constraint (c) requires

$$c_i^p(1)z_1^p(2) \leqq \sum_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^p(1)z_\alpha^p(1)\ .$$

If the machine center A_i does not process product p , $z_i^p(2)$ must be zero, and also no subsequently ordered activity will require work in process of product p from i , i.e., $\bar{a}_{\alpha i}^p(1) = 0$ for $(\alpha = i+1, i+2, \ldots, N+p)$.

Hence define

(10.6.3-10) $\mathcal{S}^{\alpha} := \{j : j \in \{1,2, \ldots, N\}$, product α is processed by $A_j\}$.

Since final products are collected starting at $\sigma = 1$, no work by an activity can be started at $\sigma = 1$. Thus

$$z_i^j(1) = E_i^j(\sigma) := 0 \quad (j = 1,2, \ldots, p) , (i = 1,2, \ldots, N) .$$

Then, for $(\sigma = 2,3, \ldots)$ the greedy solution for $z_i^p(\sigma)$ is given by $z_i^p(\sigma) = E_p^i(\sigma)$ where

$$(10.6.3\text{-}11) \quad E_p^i(2) := \begin{cases} 0 \text{ if } i \notin \mathcal{S}^p \\ \operatorname{Min}\left[\dfrac{\sum\limits_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^p(1) E_p^{\alpha}(1)}{c_i^p(1)} , \bar{\bar{z}}_i(2)\right] \text{ if } i \in \mathcal{S}^p \\ \qquad (i = 1,2, \ldots, N) . \end{cases}$$

$$(10.6.3\text{-}12) \quad E_p^i(\sigma) := \begin{cases} 0 \text{ if } i \notin \mathcal{S}^p \\ \operatorname{Min}\left[\dfrac{\sum\limits_{\tau=1}^{\sigma-1} \sum\limits_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^p(\tau) E_p^{\alpha}(\tau) - \sum\limits_{\tau=2}^{\sigma-1} c_i^p(\tau-1) E_p^{\alpha}(\tau)}{c_i^p(\sigma-1)} , \bar{\bar{z}}_i(\sigma)\right] \\ \qquad (i = 1,2, \ldots, N) . \end{cases}$$

In a similar way, using the third part of constraints (a); the greedy solution becomes $z_i^j(\sigma) = E_j^i(\sigma)$ where

$$(10.6.3\text{-}13)\quad E_j^i(2) := \begin{cases} 0 \quad \text{if} \quad i \notin \mathcal{S}^j \\ \operatorname{Min}\left[\dfrac{\sum_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^{j}(1)E_j^{\alpha}(1)}{c_i^j(1)}\,,\ \left(\bar{\bar{z}}_i(2) - \sum_{\alpha=j+1}^{p} E_\alpha^i(2)\right)\right]. \end{cases}$$

$$(10.6.3\text{-}14)\quad E_j^i(\sigma) := \begin{cases} 0 \quad \text{if} \quad i \notin \mathcal{S}^j \\ \operatorname{Min}\left[\dfrac{\sum_{\tau=1}^{\alpha-1}\sum_{\alpha=i+1}^{N+p} \bar{a}_{\alpha i}^{j}(\tau)E_j^{\alpha}(\tau) - \sum_{\tau=2}^{\sigma-1} c_i^j(\tau-1)E_j^{(i)}(\tau)}{c_i^j(\sigma-1)}\,,\right. \\ \qquad\qquad \left.\left(\bar{\bar{z}}_i(\sigma) - \sum_{\alpha=j+1}^{p} E_\alpha^i(\sigma)\right)\right]. \end{cases}$$

By calculating in reverse order of activity and reverse order of product, over all time periods in each case, one may determine a greedy production plan for the job shop.

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